FUNCTIONAL ANALYSIS

(MTH3C13)

SELF LEARNING MATERIAL

III SEMESTER (Core Course)

M.Sc. MATHEMATICS
(2019 Admission)

UNIVERSITY OF CALICUT

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MTH3C13 : FUNCTIONAL ANALYSIS

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The only way to learn Mathematics is to do Mathematics!

Paul Halmos

Chapter 1

Introduction

Functional Analysis is one of the highest abstractions in the main stream of Mathematics where generalization and abstraction not only became the most fashionable, but side by side it became unreasonably effective and unpredictably relevant. It is in this hard core area, one sees the unity and the largest interplay of the different branches of Mathematics. It is not possible to imagine any future development of Mathematics or its applications which will not bear on it the impact of the thought

processes of this area of human knowledge. 'Functional Analysis' may be thought of as the study of certain topological - algebraic structures and of the methods by which knowledge of these structures can be applied to analytic problems. It can be considered as a rare blend of the three important branches of Mathematics, namely, algebra, analysis and topology.

The idea of Functional Analsis was first introduced in 1887 by Volterra in his work on Integral equations. Early contributions to Functional analysis were made by J. Hadamard around 1903, and he also suggested the name Functional Analysis. The axioms for a normed space as is stated here goes back to the work of F. Riesz on integral equations. The beauty of a normed space, which is the basic structure considered in functional analysis, is that one can talk of unit balls and spheres even when there is no obvious Euclidean context. In fact, the purpose of functional analysis is to introduce a geometric context where there was apparently none earlier. It is the geometric context in analysis which helps us talk about approximations, limits, maximum, optimization etc., and manipulate with them as we would do in a Euclidean space. The construct Banach space, normed spaces which are complete, is one of the most important objects of study of 20th century Mathematics and has al-

ready permeated several areas of applications. It is an important tool for scientific investigations in Mathematical physics, electrical engineering, and optimization as Euclidean geometry and Trigonometry are for surveying.

The notion of inner product, which is a generalization of the dot product of vectors in Euclidean spaces, helps to introduce orthogonality in arbitrary vector spaces. Inner product spaces which are complete, are known as Hilbert spaces. The axioms for abstract Hilbert space were first enunciated by John Von Neumann (1903 - 1957) in 1927, in his monumental exposition of the mathematical formulation of Quantum Mechanics. A large area of current research interest is centered around the theory of operators on Hilbert spaces.

The concept of completeness which is crucial in Banach and Hilbert spaces, is important in the following philosophical sense. Analysis is the study of limit processes, where we are not talking of numbers alone, but functions in general. In fact, what we study could be applied to complex phenomena, available in real world or in scientific investigations. A scientific investigation make a finite number (though large) of observations and usually wants to conclude something about the ideal situation, which

is beyond the finite number of experiments. Thus from the finite to the infinite, there is a process of approximations. It is a limit process of phenomena, not of numbers. Functional analysis attempts to give a model for this limit process and in many surprising instances, it has not only succeeded in giving a model, but it has come out with such approximations which are workable. In every one of these manipulations with the process of going from the finite to infinite, one has to be, in some sense assured the existence of the limiting case, within the frame - work of the space under consideration. This is the requirement of "completeness".

Functional Analysis plays an increasing role in the applied sciences as well as in Mathematics itself. As Functional Analysis is a blend of algebra, analysis and topology, for enjoying Functional Analysis, the student should have rather a good background in these areas.

In Module I (Chapter 2), we recall some basics of linear algebra, and introduce normed spaces. In Module II (Chapter 3), we study about Hilbert spaces, and bounded linear functionals. The dual space of a normed space is introduced in Module III (Chapters 4 and 5). Also, we study about bounded linear operators and realize it as a normed space endowed with operator norm and consider different types of operators.

Chapter 2

Normed Spaces

Students are advised to revise the basic ideas of vector spaces and linear transformations that they have studied in their Linear Algebra course.

Linear algebra is the branch of Mathematics concerned with the study of vectors, vector spaces (or linear spaces) and linear transformations (or linear mappings) between vector spaces. The ideas behind the abstract notion of a vector space occurred in many concrete examples during the nineteenth century and earlier. It was Descartes and Fermat who first discussed the

vector spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ in much the way that are presented today. In the study of vector spaces, we generalize the geometric concept of a vector as a line segment of given length and direction advantageously in an abstract way. The modern definition of a vector space seems to be due to the Italian mathematician Giuseppe Peano. In Functional Analysis, we study about linear spaces endowed with some additional structures that helps to study about continuity and convergence, and some aspects of geometry.

In what follows, $\mathbb{F}$ will denote either $\mathbb{R}$, the field of real numbers or $\mathbb{C}$, the field of complex numbers.

2.1 Basics of Linear Algebra

Recall that a ***linear space*** (or a ***vector space***) over $\mathbb{F}$ is a non empty set E along with a function $+ : E \times E \to E$, called ***addition*** and a function $. : \mathbb{F} \times E \to E$, called ***scalar multiplication***, such that

- $(E, +)$ is an abelian group,
- 1. $k.(x + y) = k.x + k.y$

2. $(k+l).x = k.x + l.x$
3. $(kl)x = k.(lx)$
4. $1.x = x$, for all $x, y \in E$, and for all $k \in \mathbb{F}$.

The space $\mathbb{F}^n$ of all $n-$tuples of elements of $\mathbb{F}$ is an example for a linear space over the field $\mathbb{F}$.

Exercise 2.1. Verify that $\mathbb{F}^n$ is a linear space over the field $\mathbb{F}$.

A nonempty subset E_1 of a linear space E is said to be ***subspace*** of E if $kx + y \in E_1$ whenever $x, y \in E_1$, and $k \in \mathbb{F}$. i.e., E_1 is closed relative to the linear space operations in E, and in this case we write $E_1 \hookrightarrow E$.
The set consisting of only the zero vector of E is a subspace of E, called the ***zero subspace*** of E, and is denoted by 0.

A subspace E_1 of a linear space E is said to be a ***proper*** if $E_1 \neq E$.

Example 2.1.1. Let s be the set of all scalar sequences $(a_i)_{i=1}^{\infty}$, where $a_i \in \mathbb{F}$. Then s is a linear space with respect to the addition of sequences and multiplication with real (or complex) numbers.
The set ℓ_∞ of all bounded scalar sequences is a subspace of s.

Also, since any convergent sequence is bounded, the set c of all convergent sequences is a subspace of ℓ_∞, and the space c_0 of all sequences that converges to zero, is a subspace of c.

Let s^* be the set of all *sequences with finite support* (i.e., the set of all sequences with all but finite zero elements.) Then, a sequence $(a_i)_{i=1}^{\infty} \in s^*$ if and only if there exists $N \in \mathbb{N}$ such that $a_i = 0$ for all $i > N$. Any such sequence has only finitely many non zero terms and hence converges to 0. So s^* is a subspace of c_0. (Some authors denotes the space s^* as c_{00}.)

Thus, we have $s^* \subseteq c_0 \subseteq c \subseteq \ell_\infty \subseteq s$. □

Exercise 2.2. Show that the inclusions $s^* \subseteq c_0 \subseteq c \subseteq \ell_\infty \subseteq s$ are proper.

If E_1 is a subset of a linear space E, the ***linear span*** of E_1, denoted by ***span*** E_1 is the smallest subspace of E that contains the set E_1. Equivalently, it is the intersection of all subspaces of E containing E_1.

If E_1 is non empty, span E_1 is the set of all (finite) linear combinations of elements of E_1. i.e.,
span $E_1 = \{k_1x_1+k_2x_2+...+k_nx_n | k_1, k_2, ...k_n \in \mathbb{F}, x_1, x_2, ..., x_n \in E_1\}$.

If span $E_1 = E$, we say that E_1 ***spans*** E.

A set of vectors $x_1, x_2, ..., x_n$ of a linear space E is said to be ***linearly dependent*** if there exists numbers $a_1, a_2, ..., a_n$, not all of them are zeros, such that $a_1x_1 + a_2x_2 + ... + a_nx_n = 0$.

It is called ***linearly independent*** if it is not linearly dependent.
Thus, a set of vectors $x_1, x_2, ...x_n$ of a linear space E is linearly independent if $\sum_{i=1}^{n} a_ix_i = 0$ implies that all $a_i = 0$.

A subset B of a linear space E is called a ***basis*** for E if span $B = E$ and B is linearly independent.
Equivalently; a basis is a ***maximal*** linearly independent set.
If B is a basis for the linear space E, any element of E, can be expressed uniquely as a linear combination of elements of B (*basis vectors*).

If a linear space E has a basis consisting of a finite number of elements, then E is said to be ***finite dimensional*** and the number of elements in a basis for E is called the ***dimension*** of E.
The space $\{0\}$ is said to have ***zero*** dimension.
If a linear space contains an infinite linearly independent subset, then it is said to be ***infinite dimensional.***

Example 2.1.2. The set $\{e_1, e_2, ..., e_n\}$, in $\mathbb{F}^n$ with, $e_i = (0, 0, ..., 0, 1, 0, ...0)$, where 1 occurs only at i^{th} place, is a basis for the space $\mathbb{F}^n$ of all $n-$tuples of elements of $\mathbb{F}$, so that dimension of $\mathbb{F}^n$ is n. □

Example 2.1.3. Let $C[0, 1]$ be the set of all continuous functions defined on the interval $[0, 1]$. Then $C[0, 1]$ is a linear space with respect to the operations point wise addition and scalar multiplication. This space is infinite dimensional since the functions $p_n(t) = t^n,\ n \in \mathbb{N}$ are linearly independent. □

Let E_1 and E_2 be linear spaces over $\mathbb{F}$. A ***linear map*** from E_1 to E_2 is a function $A : E_1 \to E_2$ such that $A(ax + by) = aA(x) + bA(y)$ for all $x, y \in E_1$ and $a, b \in \mathbb{F}$.
We often write Ax instead of $A(x)$.

Two important sets are associated with such a linear transformation; namely
$\ker(A) = \{x \in E_1 \mid Ax = 0\}$, called the ***kernel*** of A, and,
$\mathrm{Im}A = \{Ax \mid x \in E_1\}$, called the ***image*** of A.

It is easy to check that $\ker A$ is a subspace of E_1, and $\mathrm{Im}(A)$ is a subspace of E_2. Also, A is one-to-one if and only if $\ker A = 0$. (Here 0 stands for the zero subspace of E_1.)

A linear map $A : E_1 \to E_2$ is an ***isomorphism*** if $\ker A = 0$ and $\mathrm{Im}A = E_2$. i.e., if and only if A is one-to-one and onto. In this case, A is invertible with $A^{-1} : E_2 \to E_1$, which is also linear.

Exercise 2.3. Show that any finite dimensional vector space is isomorphic to $\mathbb{F}^n$ for some $n \in \mathbb{N}$.

If E is a linear space over $\mathbb{F}$ and if E_1 is a subspace of E, we can define a new linear space; the ***quotient space*** of E with respect to E_1, denoted by E/E_1 as follows:

$$E/E_1 = \{[x] \mid x \in E\},$$

where $[x] = x + E_1 = \{x + y \mid y \in E_1\}$, called ***cosets*** of E relative to the subspace E_1. The vector space operations in E/E_1 are $[x] + [y] = [x + y]$ and $k[x] = [kx]$, for $x, y \in E, k \in \mathbb{F}$. The identity element for addition is $[0] = \{0 + y \mid y \in E_1\} = E_1$.

Note that $x \in [x]$ for all $x \in E$, since $x = x + 0$ and $0 \in E_1$, as it is a subspace. If $x \in E_1$, then $[x] = [0] = E_1$. Also, if $x, y \in E$, then the two cosets $[x], [y]$ either coincides or they are disjoint sets.

To see this, let $z \in [x] \cap [y]$. Then $z = x + y_1$ and $z = y + y_2$ for some $y_1, y_2 \in E_1$. Therefore, $z - x$ and $z - y$ belongs to E_1, and $y - x = (z - x) - (z - y) \in E_1$. If $a \in [x]$, then $a - x \in E_1$, so that $(a - x) - (y - x) = a - y \in E_1$ and $a = y + (a - y) \in [y]$. Thus $[x] \subseteq [y]$. Similarly, we get $[y] \subseteq [x]$. Hence, if $[x], [y]$ are not disjoint, then $[x] = [y]$.

Exercise 2.4. Show that $[x] = [y]$ if and only if $x - y \in E_1$.

The dimension of the quotient space E/E_1 is called the ***codimension*** of E_1 with respect to E and is denoted by $\text{codim}_E E_1$.

Example 2.1.4. Consider the quotient space c/c_0. (*See* Example 2.1.1) For any sequence $x = (x_n)$ in c, we can write $x + c_0 = a(1 + c_0)$, where $a = \lim_{n\to\infty} x_n$ and 1 is the constant sequence with all terms as 1. Thus, $[x] = a[1]$ for any $x \in c$, so that $\{[1]\}$ forms a basis for c/c_0 and hence $\text{codim}_c c_0 = 1$. □

A set of vectors $x_1, x_2, ..., x_n$ of a linear space E is said to be ***linearly independent relative to the subspace*** E_1 if $\sum_{i=1}^{n} a_i x_i \in E_1$ implies $a_i = 0$ for all i. Note that the set of vectors $x_1, x_2, ..., x_n$ is linearly independent if and only if it is linearly independent relative to the zero subspace.

We now use this concept to derive a characterization of dimension of E/E_1.

Lemma 2.1.5. *Let E be a linear space over $\mathbb{F}$ and E_1 be a subspace of E. Then, the dimension of E/E_1 is n if and only if there exists a set of vectors $x_1, x_2, ..., x_n$ linearly independent relative to E_1 such that for every $x \in E$, there exists a unique set of numbers $a_1, a_2, ..., a_n$ and a unique vector $y \in E_1$ such that $x = \sum_{i=1}^{n} a_i x_i + y$. Moreover, the cosets $[x_1], [x_2], ..., [x_n]$ forms a basis for E/E_1.*

Proof. Suppose that dimension of E/E_1 is n. Choose a basis $[x_1], [x_2], ..., [x_n]$ for E/E_1. We claim that the set of vectors $x_1, x_2, ..., x_n$ is linearly independent relative to E_1.

Let $\sum_{i=1}^{n} a_i x_i \in E_1$. Then,

$$\sum_{i=1}^{n} a_i [x_i] = [\sum_{i=1}^{n} a_i x_i] = [0] \Rightarrow a_i = 0, \forall i.$$

Hence, the vectors $x_1, x_2, ..., x_n$ are linearly independent relative to E_1. Also, for any $[x] \in E/E_1$,, there are numbers $a_1, a_2, ..., a_n$ such that $[x] = \sum_{i=1}^{n} a_i [x_i] = [\sum_{i=1}^{n} a_i x_i]$, implies $x = \sum_{i=1}^{n} a_i x_i + y$ for

some $y \in E_1$.

To see that this representation is unique, suppose we also have $x = \sum_{i=1}^{n} b_i x_i + z$ with $z \in E_1$. Then $\sum_{i=1}^{n}(a_i - b_i)x_i = z - y \in E_1$. Since the vectors $x_1, x_2, ..., x_n$ are linearly independent relative to E_1, this shows that $a_i = b_i, \forall i$ and $y = z$.

For the converse part, assume that for each $x \in E$, we have a unique expression $x = \sum_{i=1}^{n} a_i x_i + y$ with $y \in E_1$, and with the set of vectors $x_1, x_2, ..., x_n$ are linearly independent relative to E_1. Note that

$$\sum_{i=1}^{n} a_i[x_i] = [\sum_{i=1}^{n} a_i x_i] = [0] \Rightarrow \sum_{i=1}^{n} a_i x_i \in E_1 \Rightarrow a_i = 0, \forall i.$$

This implies that $[x_1], [x_2], ..., [x_n]$ are linearly independent in E/E_1. Also, for any $[x] \in E/E_1$, by assumption, $x = \sum_{i=1}^{n} a_i x_i + y$ with $y \in E_1$. Thus,

$$x - \sum_{i=1}^{n} a_i x_i \in E_1 \Rightarrow [x] = [\sum_{i=1}^{n} a_i x_i] = \sum_{i=1}^{n} a_i [x_i].$$

$\Rightarrow \{[x_1], [x_2], ..., [x_n]\}$ is a basis for E/E_1 and $\dim E/E_1 = n$. $\square$

2.2 Normed Spaces : Definition and Examples

In this session, we introduce the concept of ***normed spaces*** which is fundamental to the development of the theory of ***Banach spaces***, which plays a key role in functional analysis. The notion of normed spaces can be thought of as a generalization of the n-dimensional space $\mathbb{F}^n$ with the Euclidean length given by $(\sum_{j=1}^{n} x_i^2)^{\frac{1}{2}}$. The concept of norm in a linear space can be used to define a metric and hence the notion of convergence in a linear space, which is necessary, for example to give a meaning to the sum of an infinite series.

Also we shall see that every normed space is a metric space and therefore a topological space; thus it is natural that the topological concepts such as open subset, closed subset, limit, closure, denseness, compactness, separability, etc., make sense on a normed space. Among the class of normed spaces, the most important ones are the ***complete normed spaces*** which are universally known as Banach spaces.

Recall that a ***metric*** d on a nonempty set E is a function $d : E \times E \to \mathbb{R}$ such that for all $x, y, z \in E$,

- $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$ (Triangle inequality .)

A ***metric space*** is a non empty set E along with a metric on it.

For example, the modulus or absolute value $|\cdot|$ defines a metric on $\mathbb{F}$, namely $d(x, y) = |x - y|$, for $x, y \in \mathbb{F}$.

Definition 2.2.1. Let E be a linear space over $\mathbb{F}$. A ***norm*** on E is a function $||.||$ from E to $\mathbb{R}$ such that for all $x, y \in E$ and for all $\lambda \in \mathbb{F}$,

1. $||x|| \geq 0$ and $||x|| = 0$ if and only if $x = 0$. - [Positive definiteness]
2. $||\lambda x|| = |\lambda|\ ||x||$ - [Homogeneity of norm]
3. $||x + y|| \leq ||x|| + \ ||y||$ - [Triangle inequality]

A ***normed space*** is a linear space with a norm on it.

We denote a linear space E equipped with a norm $||\cdot||$ as X, i.e., $X = (E, ||\cdot||)$.

In a normed space X, we define a metric by $d(x, y) = ||x-y||$ for $x, y \in X$.

Exercise 2.5. Verify that d is a metric on X.

Definition 2.2.2. Let $||\cdot||_1$ and $||\cdot||_2$ be any two norms on a linear space E. Then the norm, $||\cdot||_1$ is said to be ***stronger*** than $||\cdot||_2$ if there exists a constant $C > 0$ such that $||x||_2 \leq C\ ||x||_1$ for all $x \in E$.

The norms $||\cdot||_1$ and $||\cdot||_2$ are said to be ***equivalent*** if there exist constants $C, c > 0$ such that $c\ ||x||_2 \leq ||x||_1 \leq C\ ||x||_2$ for all $x \in E$.

Exercise 2.6. Prove that any subspace Y of a normed space X is again a normed space with the inherited norm (obtained by restricting the norm in X to the space Y).

Example 2.2.3. On the linear space ℓ_∞ of all bounded scalar sequences, define $||\cdot||_\infty$ as the supremum of the absolute values of the terms of the sequences; i.e., $||x||_\infty = \sup_i |a_i|$ for any $x = (a_i)_{i=1}^\infty \in \ell_\infty$. It is easy to verify that this is a norm on ℓ_∞.

By Exercise 2.6, it follows that s^*, c_0, c are all normed spaces with this norm. □

Example 2.2.4. Consider $C[0,1]$, the space of all continuous functions defined on the interval $[0,1]$. For $f \in C[0,1]$, define $||f||_\infty = \sup_{t\in[0,1]} |f(t)|$. This makes $C[0,1]$ a normed space. The space of all polynomial functions $P[0,1]$ is a subspace of this normed space. □

Exercise 2.7. Verify that $||f||_\infty = \sup_{t\in[0,1]} |f(t)|$ is a norm on the space $C[0,1]$.

Before listing more examples for normed spaces, we now state and prove two important inequalities: Hölder's inequality and Minkowski's inequality.

Theorem 2.2.5. ***Hölder's inequality for Sequences***
For every sequence of scalars (a_i) and (b_i) and for $1 < p < \infty$, we have

$$|\sum a_k b_k| \leq \sum |a_k b_k| \leq (\sum |a_k|^p)^{\frac{1}{p}} (\sum |b_k|^q)^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. (The case of this inequality with $p = q = 2$ is known as the Cauchy-Schwartz inequality.)

Proof. If is enough to prove this inequality for every finite set of numbers $(a_n)_{n=1}^N$ and $(b_n)_{n=1}^N$, because by letting $N \to \infty$ in the finite case, we see that the above inequality holds for countable sets too.
From $\frac{1}{p} + \frac{1}{q} = 1$, we get $p + q = pq \Rightarrow p = (p-1)q$ and $\frac{p}{p-1} = q$, subtracting 1 from both sides, we obtain $\frac{1}{p-1} = q - 1$.

Set $c_i = |a_i|/(\sum |a_j|^p)^{\frac{1}{p}}$ and $d_i = |b_i|/(\sum |b_j|^q)^{\frac{1}{q}}$, then $\sum c_i^p = 1, \sum d_i^q = 1$, and to prove the Hölder's inequality, we have to show that $\sum c_i d_i \leq 1$. Consider the function $y = x^{p-1}$ and integrate it with respect to x from $x = 0$ to $x = c_i$. This integral represents the area S_1 in the figure 2.1. Similarly, integrating the inverse $x = y^{1/(p-1)} = y^{q-1}$ from $y = 0$ to $y = d_i$, we get the area S_2 in the figure 2.1.

Also, from the figure 2.1, it is clear that the sum of the areas $S_1 + S_2$ always exceeds the area of the rectangle,namely $c_i d_i$. Thus, $c_i d_i \leq S_1 + S_2 = \int_0^{c_i} x^{p-1} dx + \int_0^{d_i} y^{q-1} dy = \frac{1}{p} c_i^p + \frac{1}{q} d_i^q$, with equality only if $d_i = c_i^{p-1}$.

Taking sum of these inequalities on i, we get $\sum c_i d_i \leq \frac{1}{p} \sum c_i^p + \frac{1}{q} \sum d_i^q \leq 1$, since $\sum c_i^p = 1$ and $\sum d_i^q = 1$. This completes the proof. □

Remark 2.2.6. By formally defining $(\sum |b_k|^q)^{\frac{1}{q}}$ to be $\sup |b_k|$

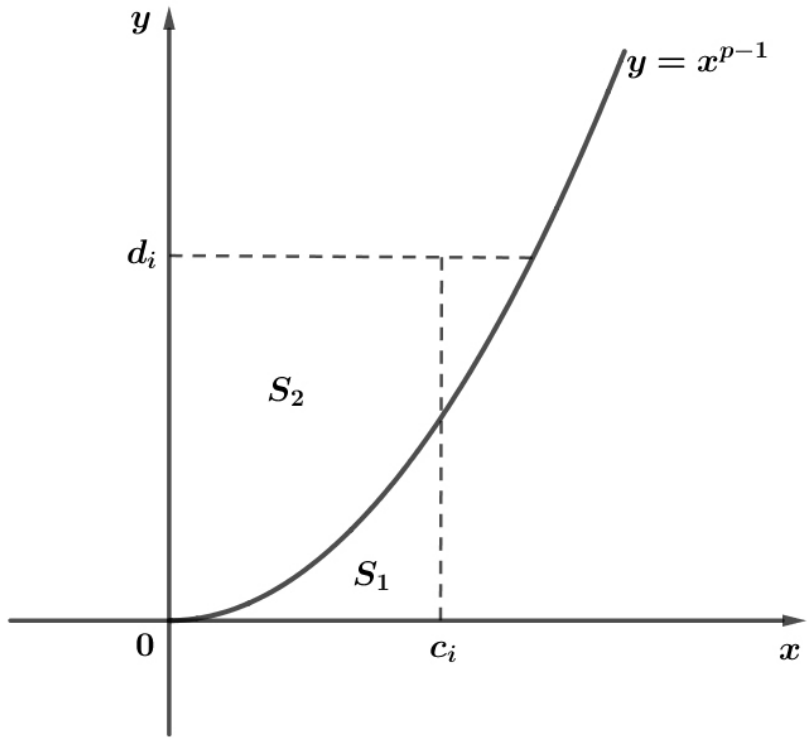

Figure 2.1: $c_i d_i \leq S_1 + S_2$

for the case $q = \infty$, we can interpret the Hölder's inequality for all p with $1 \leq p \leq \infty$.

Theorem 2.2.7. ***Hölder's inequality for Functions***
Let $1 < p < \infty$ and let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, for all functions f, g on an interval $[a, b]$ such that the integrals $\int_a^b |f(t)|^p dt, \int_a^b |g(t)|^q dt$ and $\int_a^b |f(t)g(t)|dt$ exist (as Riemann integrals), we have

$$\int_a^b |f(t)g(t)|dt \leq (\int_a^b |f(t)|^p dt)^{1/p} (\int_a^b |g(t)|^q dt)^{1/q}.$$

Remark 2.2.8. If $\int_a^b f(t)g(t)dt$ also exists, then from the inequality, $|\int_a^b f(t)g(t)dt| \leq \int_a^b |f(t)g(t)|dt$, it follows that

$$|\int_a^b f(t)g(t)dt| \leq (\int_a^b |f(t)|^p dt)^{1/p} (\int_a^b |g(t)|^q dt)^{1/q}.$$

Proof. Consider a partition $a = x_0 < x_1 < ... < x_n = b$ of the interval $[a, b]$ into n equal subintervals of length $\Delta x = \frac{b-a}{n}$. We

have, by using the Hölder's inequality for sequences,

$$\begin{aligned}\sum_{1=1}^{n}|f(x_i)g(x_i)|\Delta x &= \sum_{1=1}^{n}|f(x_i)g(x_i)|(\Delta x)^{\frac{1}{p}+\frac{1}{q}}\\ &= \sum_{1=1}^{n}(|f(x_i)|^p\Delta x)^{\frac{1}{p}}(|g(x_i)|^q\Delta x)^{\frac{1}{q}}\\ &\leq (\sum_{1=1}^{n}|f(x_i)|^p\Delta x)^{\frac{1}{p}}(\sum_{1=1}^{n}|g(x_i)|^q\Delta x)^{\frac{1}{q}}\end{aligned}$$

As $n \to \infty$, the sums will be replaced by the integrals, and we get the desired inequality. □

Using the Hölder's inequality, we now prove the Minkowski's inequality.

Theorem 2.2.9. ***Minkowski's inequality for Sequences*** *For every sequence of scalars $a = (a_i)$ and $b = (b_i)$ and for $1 \leq p \leq \infty$, we have*

$$\|a + b\|_p \leq \|a\|_p + \|b\|_p,$$

where, $\|a\|_p = (\sum |a_i|^p)^{\frac{1}{p}}$ for $1 < p < \infty$, and $\|a\|_\infty = \sup_i |a_i|$.

Proof. The case $p = 1$ follows from the triangle inequality $|a_i +$

$b_i| \leq |a_i| + |b_i|$ for all $a_i, b_i \in \mathbb{F}$. The case $p = \infty$ also follows similarly from the subadditivity of the maximum (supremum)function.

To prove the case $1 < p < \infty$, we have,

$$\begin{aligned}
\|a+b\|_p^p &= \sum |a_i + b_i|^p \\
&\leq \sum (|a_i + b_i|)^{p-1}(|a_i| + |b_i|) \\
&= \sum (|a_i + b_i|)^{p-1}|a_i| + \sum (|a_i + b_i|)^{p-1}|b_i| \\
&\leq (\sum (|a_i + b_i|)^p)^{\frac{1}{q}} (\sum (|a_i|)^p)^{\frac{1}{p}} \\
&\qquad + (\sum (|a_i + b_i|)^p)^{\frac{1}{q}} (\sum (|b_i|)^p)^{\frac{1}{p}} \\
&= (\sum (|a_i + b_i|)^p)^{\frac{1}{q}} (\|a\|_p + \|b_p\|)
\end{aligned}$$

where the first inequality follows from the triangle inequality of modulus and the last inequality is obtained by applying the Hölder's inequality, and noting that $(p-1)q = p$. Thus,

$$\|a+b\|_p^p = \sum |a_i + b_i|^p \leq (\sum (|a_i + b_i|)^p)^{\frac{1}{q}} (\|a\|_p + \|b_p\|).$$

Dividing both sides by $(\sum (|a_i + b_i|)^p)^{\frac{1}{q}}$, we get

$$(\|a+b\|_p^p)^{1-1/q} \leq \|a\|_p + \|b_p\| \Rightarrow \|a+b\|_p \leq \|a\|_p + \|b_p\|,$$

since $1 - \frac{1}{q} = \frac{1}{p}$. □

Theorem 2.2.10. ***Minkowski's inequality for Functions*** *Let $1 < p < \infty$ and let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, for all functions f, g on an interval $[a, b]$ such that the integrals $\int_a^b |f(x)|^p dx, \int_a^b |g(x)|^p dx$ exist, the integral $\int_a^b |f(x) + g(x)|^p dx$ exists too, and it satisfies*

$$(\int_a^b |f(x)+g(x)|^p dx)^{1/p} \leq (\int_a^b |f(x)|^p dx)^{1/p} + (\int_a^b |g(x)|^p dx)^{1/p}.$$

Proof. Consider a partition $a = x_0 < x_1 < ... < x_n = b$ that divides the interval $[a, b]$ into n equal subintervals of length $\Delta x = \frac{b-a}{n}$. We have, by using the Minkowski's inequality for sequences,

$$(\sum_{i=1}^{n} |f(x_i)+g(x_i)|^p \Delta x)^{1/p} \leq (\sum_{i=1}^{n} |f(x_i)|^p \Delta x)^{1/p} + (\sum_{1=1}^{n} |g(x_i)|^p \Delta x)^{1/p}$$

As $n \to \infty$, the sums will be replaced by the integrals, and we get the desired inequality. □

Example 2.2.11. For $1 \leq p < \infty$, let ℓ_p^n denotes the linear space of all n-tuples of real numbers equipped with the norm $\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p}$, where $x = (x_1, x_2, ..., x_n)$, i.e., $\ell_p^n = (\mathbb{R}^n, \|\cdot$

$\|_p$). Here, the triangle inequality of the norm follows from the Minkowski's inequality. Similarly, ℓ_∞^n denotes the normed space $(\mathbb{R}^n, \|\cdot\|_\infty)$, where $\|x\|_\infty = \max_i |x_i|$.

Also, for $1 \leq p < \infty$, let ℓ_p denotes the linear space of all p- summable sequences of real numbers (i.e., all sequences $x = (x_1, x_2, ..., x_n, ...)$, for which $\sum_{i=1}^{\infty} |x_i|^p < \infty$.) equipped with the norm $\|x\|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p}$. i.e., $\ell_p = (\mathbb{R}^\infty, \|\cdot\|_p)$. The facts that this space ℓ_p is closed under addition and the triangle inequality of the norm $\|\cdot\|_p$ follows from the Minkowski's inequality. □

Exercise 2.8. If $1 \leq p < q < \infty$, prove that $\ell_p \subsetneq \ell_q$.

The norm on a linear space induces a metric and that helps us to introduce topological and geometric notions.

We say that a sequence (x_n) in a normed space X converges to a point $x \in X$ if and only if $\|x_n - x\| \to 0$, as $n \to \infty$.

From the triangle inequality of norm, it follows that norm is a continuous function, i.e., if $x_n \to x$ in X, then $\|x_n\| \to \|x\|$. Using triangle inequality of norm, one have $|\,\|x\| - \|y\|\,| \leq \|x - y\|$, and this in fact, shows that norm is uniformly continuous.

Let X be a normed space, $x_0 \in X$ and $r > 0$. The set

$D_r(x_0) = \{x \mid \|x - x_0\| < r\}$ is said to be the ***open ball*** with *center* at $x = x_0$ and of *radius* r.

A set $O \subseteq X$ is said to be ***open*** if and only if for every $x \in O$, $\exists r > 0$ such that $D_r(x) \subseteq O$.

A subset F of X is said to be ***closed*** if for every sequence $x_n \in F$ that converges to some $x \in X$, implies that $x \in F$.

Lemma 2.2.12. *A set O in a normed space X is open if and only if its complement $F = O^c$ is closed.*

Proof. Let O be an open set in X. Let $F = O^c$ and let (x_n) be a sequence in F and $x_n \to x \in X$. Suppose $x \in O$. Since O is open, $\exists r > 0$ such that the open ball $D_r(x) \subseteq O$. Since $x_n \to x$, for large n, $\|x_n - x\| < r \Rightarrow x_n \in O$, which is a contradiction, since $x_n \in F = O^c$. This proves that $x \in F \Rightarrow F$ is closed.

Conversely assume that F is closed and $O = F^c$. Let $x \in O$. We claim that $\exists \epsilon > 0$ such that $D_\epsilon(x) \subseteq O$. If not, for each decreasing sequence $\epsilon_n > 0$, converging to 0, $D_{\epsilon_n} \cap F$ is non-empty, so that $\exists x_n \in F$ such that $\|x_n - x\| < \epsilon_n \Rightarrow x_n \to x$. Since F is closed, we get $x \in F$. This contradiction shows that $D_\epsilon(x) \subseteq O$. Since $x \in O$ is arbitrary, this implies O is open. □

Let X be a normed space and $A \subseteq X$. We define the ***closure*** of A as the set consisting of all $x \in X$ such that there exists a sequence (x_n) in A with $x_n \to x$, and is denoted by $\overline{A}$. For any $x \in A$, we have the constant sequence $x_n = x, \forall n$, converging to x, so that $A \subseteq \overline{A}$. Note that A is closed if and only if $\overline{A} = A$.

We say that A is a ***dense*** set in X, if $\overline{A} = X$.

Exercise 2.9. In any normed space X, show that the empty set and the whole set X are both open and closed.

Exercise 2.10. In any normed space X, show that union of any arbitrary collection of open sets and intersection of any finite collection of open sets are open.

Exercise 2.11. Show that $\overline{A}$ is the intersection of all closed sets containing A.

Let X be a normed space and $x, y \in X$. Then the set of all points $\{\lambda x + (1 - \lambda)y \mid 0 \leq \lambda \leq 1\}$ is geometrically the line segment joining x and y. We call this set an ***interval*** and is denoted by $I[x, y]$.

A subset M of a linear space E is said to be ***convex*** if and only if for every two points $x, y \in M$, the interval $I[x, y]$ is contained in M.

Exercise 2.12. Show that intersection of a family of convex sets in a linear space is convex.

If E is a normed space, the set

$$\mathcal{D}(E) = \{x \in E \mid \|x\| \leq 1\},$$

is called the ***unit ball*** of the space E. Since $\|x\| = \|-x\|$, we see that $x \in \mathcal{D}(E)$ if and only if $-x \in \mathcal{D}(E)$, i.e., the unit ball is ***symmetric with respect to the origin.***

The unit ball $\mathcal{D}(E)$ is a convex set. To see this, let $x, y \in \mathcal{D}(E)$. Then for any $0 \leq \lambda \leq 1$,

$$\|\lambda x + (1-\lambda)y\| \leq \lambda\|x\| + (1-\lambda)\|y\| \leq \lambda + (1-\lambda) = 1.$$

Therefore, $I[x, y] \subseteq \mathcal{D}(E)$. $\Rightarrow \mathcal{D}(E)$ is convex.

The set $\mathcal{S}(E) = \{x \in E \mid \|x\| = 1$ is called the ***unit sphere*** of E.

A subset M of a normed space E is said to be ***bounded*** if there exists $C > 0$ such that $\|x\| \leq C$, $\forall x \in M$. Equivalently, $M \subset C \cdot \mathcal{D}(E)$.

In Exercise 2.6, we have noted that any subspace of a normed

space is again a normed space. Now we will discuss how to define a norm on a quotient space.

Lemma 2.2.13. *Let X be a normed space and let X_0 be a closed subspace of X. For any $[x] \in X/X_0$, define $\|[x]\| = \inf_{y\in X_0} \|x-y\|$. This is a norm on X/X_0, called the quotient norm. (Equivalently,$\|[x]\| = \inf_{y\in X_0} \|x+y\|$, since X_0 is a subspace.)*

Proof. Clearly, $\|[x]\| \geq 0$, and if $[x] = 0$, then $x \in X_0$, so that

$$\|[x]\| = \inf_{y\in X_0} \|x-y\| = \inf_{z\in X_0} \|z\| = 0.$$

Now, if $\|[x]\| = 0$, then $\inf_{y\in X_0} \|x-y\| = 0$, so that for any $n \in \mathbb{N}$, $\exists y_n \in X_0$ with $\|x-y_n\| < \frac{1}{n} \Rightarrow y_n \to x$.
Since X_0 is closed, this implies $x \in X_0$, so that $[x] = 0$.

Also, for any number λ,

$$\begin{aligned}
\|\lambda[x]\| &= \|[\lambda x]\| \\
&= \inf_{y\in X_0} \|\lambda x - y\| \\
&= |\lambda| \inf_{y\in X_0} \|x - (y/\lambda)\| \\
&= |\lambda| \inf_{z\in X_0} \|x - z\| \\
&= |\lambda| \, \|[x]\|.
\end{aligned}$$

Now it remains to prove the triangle inequality.
Let $[x], [y] \in X/X_0$ and $\epsilon > 0$.
By definition of quotient norm, $\exists z_1, z_2 \in X_0$ such that $\|x+z_1\| \leq \|[x]\| + \epsilon$ and $\|y + z_2\| \leq \|[y]\| + \epsilon$. Therefore,

$$\begin{aligned} \|[x] + [y]\| &= \inf_{z \in X_0} \|x + y + z\| \\ &\leq \|x + y + z_1 + z_2\| \\ &\leq \|x + z_1\| + \|y + z_2\| \\ &\leq \|[x]\| + \|[y]\| + 2\epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, this proves the triangle inequality. Thus, the quotient space X/X_0 is a normed space with this norm. □

A function p on a linear space E is called a ***seminorm*** if it satisfies the properties of the norm except that it may be zero for non-zero vectors.
More precisely, a seminorm is a function $p : E \to \mathbb{R}^+ \cup \{0\}$ such that

- $p(0) = 0$,
- $p(\lambda x) = |\lambda| p(x)$,

- $p(x+y) \leq p(x) + p(y)$,

for all $x, y \in E$ and all $\lambda \in \mathbb{F}$.

Suppose p is a seminorm on a linear space E. Let E_0 be its ***kernel***, i.e., $E_0 = \{x \in E \mid p(x) = 0\}$. Let $x, y \in E_0$ and $\lambda \in \mathbb{F}$. Then $p(\lambda x + y) \leq |\lambda| p(x) + p(y) = 0$. $\Rightarrow E_0$ ***is a subspace of*** E.

Also, note that $p(x+y)$ is independent of $y \in E_0$, i.e., $p(x + y_1) = p(x + y_2)$ for any $y_1, y_2 \in E_0$.
To see this,

$$\begin{aligned} p(x+y_1) &= p(x + y_2 - y_2 + y_1) \\ &\leq p(x+y_2) + p(y_1 - y_2) \\ &= p(x+y_2) \end{aligned}$$

Similarly,

$$\begin{aligned} p(x+y_2) &= p(x + y_2 - y_1 + y_1) \\ &\leq p(x+y_1) + p(y_2 - y_1) \\ &= p(x+y_1). \end{aligned}$$

Thus, p can be regarded as a function on cosets, i.e., $p([x]) =$

$p(x)$ and it does not depend on the representative of the coset. So, a seminorm p on E defines naturally a norm on E/E_0.

It is easy to see that the space of all continuous functions $C_p[a,b]$ on an interval $[a,b]$ is a normed space with norm $\|f\|_p = (\int_a^b |f(x)|^p dx)^{1/p}$, where $1 \leq p < \infty$.

Exercise 2.13. Show that for $1 \leq p < \infty$, $\|f\|_p = (\int_a^b |f(x)|^p dx)^{1/p}$ defines a norm on the space of all continuous functions on an interval $[a,b]$.

Now consider the set,

$$R_p[a,b] = \{f : [a,b] \to \mathbb{F} \mid \int_a^b |f(x)|^p dx)^{1/p} < \infty\}.$$

For $f \in R_p[a,b]$ define $\|f\|_p = (\int_a^b |f(x)|^p dx)^{1/p}$. Then this satisfies all the requirements for a norm except, $\|f\|_p = 0 \Rightarrow f = 0$. In fact, $\int_a^b |f(x)|^p dx = 0$ only implies $f = 0$ a.e. on $[a,b]$. So, $\|\cdot\|_p$ defines a seminorm on $R_p[a,b]$. By passing to a quotient space; quotient with respect to the set of zeroes of $\|\cdot\|$, we get a normed space, denoted by $L_p[a,b]$, but its elements are not functions, but every element of this space is a class of functions. Any two representatives, say f_1, f_2, of the same class, will differ only on a "nonessential set" i.e., on a set of measure zero. We

often regard elements of this space as functions, by selecting a convenient representative of each coset.

2.3 Banach Spaces

A sequence (x_n) in a normed space X is called a ***Cauchy sequence*** if and only if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $\|x_n - x_m\| < \varepsilon, \forall n, m > N$. i.e., $\|x_n - x_m\| \to 0$ as $n > m \to \infty$.

If (x_n) is a Cauchy sequence in a normed space X, then $(\|x_n\|)$ is a Cauchy sequence in $\mathbb{R}$ and hence $\lim_{n\to\infty} \|x_n\|$ exists. (This follows from the inequality $| \|x\| - \|y\| | \leq \|x - y\|$ and from the completeness of $\mathbb{R}$.)

Recall that a sequence (x_n) in a normed space X is bounded if there exists $M > 0$ such that $\|x_n\| \leq M, \ \forall n \in \mathbb{N}$. Since every convergent sequence of real numbers is bounded, it follows that any Cauchy sequence in X is bounded.

Definition 2.3.1. A normed space X is called ***complete*** if and only if every Cauchy sequence (x_n) in X converges to an element x of the space X.

A complete normed space is called a ***Banach space.***

Example 2.3.2. From our studies of real and complex analysis in degree classes, we recall and note that the spaces $\mathbb{R}$ and $\mathbb{C}$ are Banach spaces under the norm 'modulus'. Similarly the spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ with euclidean norm $\|x\| = (\sum_{k=1}^{n} |x_k|^2)^{1/2}$, where $x = (x_1, x_2, ..., x_n)$, are Banach spaces.

The space $(\mathbb{Q}, |\cdot|)$ is not complete, for instance, the finite decimal truncations $1,\ 1.4,\ 1.41, 1.414,\ 1.4142, ...$ of $\sqrt{2}$ is a Cauchy sequence, but not convergent in $\mathbb{Q}$. □

Example 2.3.3. Consider the space $C[a, b]$ of all continuous, complex valued functions equipped with the norm (*See* Example 2.2.4)

$$\|f\|_\infty = \sup_{t\in[a,b]} |f(t)|, \quad \text{for } f \in C[a, b].$$

Let (f_n) be a Cauchy sequence in $C[a, b]$ and $\varepsilon > 0$.
Then $\exists N(\varepsilon) \in \mathbb{N}$ such that $\|f_n - f_m\|_\infty < \varepsilon, \forall n > m > N(\varepsilon)$.
$\Rightarrow |f_n(t) - f_m(t)| < \varepsilon, \forall n > m > N(\varepsilon)$.
$\Rightarrow (f_n(t))$ is a Cauchy sequence of numbers and hence converges, for any $t \in [a, b]$.
Let $f(t) = \lim_{n\to\infty} f_n(t),\ \forall t \in [a, b]$.

Then, letting $m \to \infty$ in $\|f_n - f_m\|_\infty < \varepsilon$, we get

$$\|f_n - f\|_\infty < \varepsilon, \forall n > N(\varepsilon).$$

Note that convergence with respect to this norm is the uniform convergence. Since uniform limit of a sequence of continuous functions is continuous, it follows that $f \in C[a, b]$. This shows that $(C[a, b], \|\cdot\|_\infty)$ is a complete normed space. □

Example 2.3.4. For $1 \leq p < \infty$, consider the linear space ℓ_p of all p- summable sequences of real (or complex) numbers, (i.e., all sequences $x = (x_1, x_2, ..., x_n, ...)$, for which $\sum_{i=1}^{\infty} |x_i|^p < \infty$,) equipped with the norm $\|x\|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p}$. i.e., $\ell_p = (\mathbb{R}^\infty, \|\cdot\|_p)$. (*See* Example 2.2.11)

Let $x^n = (x_m^n)_{m=1}^\infty$ be a Cauchy sequence in ℓ_p and $\varepsilon > 0$. (Here, each x^n is a sequence, $x^n = (x_1^n, x_2^n, x_3^n, ...)$.)
Then $\exists N(\varepsilon) \in \mathbb{N}$ such that $\|x^n - x^k\|_p < \varepsilon, \forall n > k > N(\varepsilon)$.
Thus, $\forall n > k > N(\varepsilon)$

$$\sum_{m=1}^{\infty} |x_m^n - x_m^k|^p \quad < \quad \varepsilon^p \tag{2.1}$$

$\Rightarrow |x_m^n - x_m^k| < \varepsilon,\ \forall n > k > N(\varepsilon)$ for each m.
This shows that for each m, (x_m^n) is a Cauchy sequence in $\mathbb{R}$(or $\mathbb{C}$) and hence converges.
Let $x_m = \lim_{n\to\infty}(x_m^n)$, for $m = 1, 2, ...$ and $x = (x_m)_{m=1}^{\infty}$.
We claim that $x \in \ell_p$ and $x^n \to x$ in ℓ_p as $n \to \infty$.
For each $M \in \mathbb{N}$, $\sum_{m=1}^{M} |x_m|^p = \sum_{m=1}^{M} (\lim_{n\to\infty} |x_m^n|^p) = \lim_{n\to\infty} \sum_{m=1}^{M} |x_m|^p \leq \sup_n \|x^n\|_p^p < C < \infty$, for some $C > 0$, since Cauchy sequences are bounded. Since this is true for any $M \in \mathbb{N}$, letting $M \to \infty$, we get $\sum_{m=1}^{\infty} |x_m|^p < \infty \Rightarrow x \in \ell_p$.

From equation 2.1, for any $M \in \mathbb{N}$, we have, for all sufficiently large n, k,

$$\sum_{m=1}^{M} |x_m^n - x_m^k|^p < \varepsilon^p$$

Letting $k \to \infty$ first and then $M \to \infty$, we get

$$\sum_{m=1}^{\infty} |x_m^n - x_m^k|^p \leq \varepsilon^p$$

This means, $\|x^n - x\|_p \leq \varepsilon$ for large values of n.
$\Rightarrow x^n \to x$ in ℓ_p.

Thus, $(\ell_p, \|\cdot\|_p)$ is a Banach space. □

Exercise 2.14. Show that $\ell_p^n = (\mathbb{R}^n, \|\cdot\|_p)$, $1 \leq p < \infty$ is a Banach space.

Exercise 2.15. Show that every finite dimensional normed linear space is a Banach space.

Exercise 2.16. Show that the space $(\ell_\infty, \|\cdot\|_\infty)$ (*See* Example 2.2.3) is a Banach space.

Suppose that E be a Banach space and E_1 be a subspace of E. Then E_1 ***is a Banach space if and only if*** E_1 ***is closed.***

To see this, first assume that E_1 is a Banach space with the inherited norm. If (x_n) is a sequence in E_1 that converges to some $x \in E$, then (x_n) is a Cauchy sequence in E_1. Since E_1 is complete, $x \in E_1$. This implies, E_1 is closed.

Conversely assume that E_1 is closed. If (x_n) is a Cauchy sequence in $E_1 \subseteq E$, it converges to some $x \in E$, since E is complete. But E_1 is closed, so that $x \in E_1$. Thus, E_1 is a Banach space.

From Exercise 2.16, it follows that the subspaces c_0 and c are Banach spaces, as they are closed in ℓ_∞.

To see c_0 is closed in ℓ_∞, let $(x^{(m)})$ be a sequence in c_0, $\varepsilon > 0$ and suppose $\|x^{(m)} - x\|_\infty \leq \varepsilon/2$ for all $m \geq N$ for some $N \in \mathbb{N}$. Since $x^{(N)} \in c_0$, $\lim_{k\to\infty} x_k^{(N)} = 0$, so that $|x_k^{(N)}| < \varepsilon/2$ for all $k > N_1$ for some $N_1 \in \mathbb{N}$. Hence for all $k > N_1$,

$$|x_k| \leq |x_k - x_k^{(N)}| + |x_k^{(N)}| \leq \|x - x^{(N)}\|_\infty + |x_k^{(N)}| < \varepsilon.$$

This implies that $\lim_{k\to\infty} x_k = 0. \ \Rightarrow \ x = (x_k) \in c_0$.

Exercise 2.17. Show that c is a closed subspace of ℓ_∞.

Exercise 2.18. Show that the space of all polynomial functions $(P[0,1], \|\cdot\|_\infty)$ is not a Banach space.

Exercise 2.19. Let X be a Banach space and E be a closed subspace of X. Show that the quotient space X/E is a Banach space.

Example 2.3.5. Consider the subspace s^* of ℓ_∞ consisting of sequences with all but finite zero terms, as a subspace of ℓ_∞.(*See* Example 2.1.1). The sequence $(a^{(n)})$ defined by,

$a^{(n)} = (1, 1/2, 1/3, ..., 1/n, 0, 0, ...)$ for $n \in \mathbb{N}$,

is a sequence in s^* and $(a^{(n)})$ converges to the sequence $(1/n)$, but it does not belongs to s^*. Thus, s^* is not closed, and hence *not* a Banach space. □

Definition 2.3.6. Let E, F be normed spaces. We say that a linear map $T : E \to F$ is an ***isometry into*** if and only if $\|Tx\|_F = \|x\|_E$, for all $x \in E$.

Next, we show that any normed space can be densely embedded in a Banach space.

Theorem 2.3.7. *Let E be a normed space. There exists a complete normed space $\hat{E}$ and a linear map $T : E \to \hat{E}$ such that $\|Tx\| = \|x\|$ for all $x \in E$, (isometry into); and ImT $(:= TE)$ is a dense set in $\hat{E}$ (i.e., $\overline{TE} = \hat{E}$.)*
We call such a space $\hat{E}$ the **completion** *of E.*

Proof. Let $\mathcal{E}$ be the space of all Cauchy sequences $\overline{x} = (x_i)_{i=1}^{\infty}$ in E. Define $p(\overline{x}) = \lim_{i\to\infty} \|x_i\|$, where $\overline{x} \in \mathcal{E}$. Note that p is well-defined (Since, if (x_n) is a Cauchy sequence in a normed space X, then $(\|x_n\|)$ is a Cauchy sequence in $\mathbb{R}$ and hence $\lim_{n\to\infty} \|x_n\|$ exists.) and is a seminorm on $\mathcal{E}$.

Let $N = \{\overline{x} \mid p(\overline{x}) = 0\}$. Then, N is the subspace of all sequences in E that converges to 0. Also, p defines a norm on the quotient space $\hat{E} = \mathcal{E}/N$, by the same formula $p(\overline{x}) = \lim_{i\to\infty} \|x_i\|$, for any representative $\overline{x}$ of an equivalence class $[\overline{x}] = \overline{x} + N \in \mathcal{E}/N$.

Define $T : E \to \hat{E}$ by $T(x) = [\overline{x}]$, where $\overline{x}$ is the constant sequence $(x, x, x, ...)$. Note that a constant sequence is a Cauchy sequence and $p(\overline{x}) = \|x\|$. Clearly, T is a linear map and $\|Tx\| = \|x\|$.

Now, we show that TE is dense in $\hat{E}$. Let $\overline{x} = (x_n) \in \mathcal{E}$. For any $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\|x_n - x_m\| \leq \varepsilon$ for all $m > n \geq N$. For such an $n > N$, define $\overline{x}_n$ as the constant sequence $(x_n, x_n, x_n, ...)$. Then $\overline{x}_n \in \mathcal{E}$, and $Tx_n = [\overline{x}_n]$. $\Rightarrow [\overline{x}_n] \in TE$.
Also, $d([\overline{x}], [\overline{x}_n]) = p([\overline{x}] - [\overline{x}_n]) = p(\overline{x} - \overline{x}_n) = \lim_{m\to\infty} \|x_m - x_n\| \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this shows that any $[\overline{x}] \in \hat{E}$ can be approximated by elements of TE. This proves that TE is dense in $\hat{E}$.

Next, we prove that $\hat{E}$ is complete. Consider a Cauchy sequence $([\overline{x}^{(n)}])$ in $\hat{E}$.
Then $p([\overline{x}^{(n)}] - [\overline{x}^{(m)}]) = p(\overline{x}^{(n)} - \overline{x}^{(m)}) \to 0$ as $n \geq m \to \infty$.
Choose $\varepsilon_n > 0$ and converging to 0, then since TE is dense in $\hat{E}$, we get $x_n \in E$ such that $p(\overline{x}^{(n)} - Tx_n) < \varepsilon_n$. We claim that

$\overline{x}^0 = (x_n)$ is a Cauchy sequence in E. We have,

$$\begin{aligned} \|x_n - x_m\| &= p(Tx_n - Tx_m) \\ &\leq p(Tx_n - \overline{x}^{(n)}) + p(\overline{x}^{(n)} - \overline{x}^{(m)}) + p(\overline{x}^{(m)} - Tx_m) \\ &\to 0 \text{ as } n \geq m \to \infty. \end{aligned}$$

Therefore, $\overline{x}^0 = (x_n) \in \mathcal{E}$.
Also,

$$\begin{aligned} p([\overline{x}^{(0)}] - [\overline{x}^{(n)}]) &= p(\overline{x}^{(n)} - \overline{x}^{(0)}) \\ &\leq p(\overline{x}^{(n)} - Tx_n) + p(Tx_n - \overline{x}^{(0)}) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus, $[\overline{x}^{(0)}] = \lim_{n\to\infty} [\overline{x}^{(n)}]$. Hence, any Cauchy sequence $([\overline{x}^{(n)}])$ in $\hat{E}$ converges in $\hat{E}$. This completes the proof. □

The completion of a normed space is unique in the following sense.

Theorem 2.3.8. *Let E be a normed space, and $X_i, i = 1, 2$ be complete normed spaces. Let $T_i : E \to X_i$ be isometries into dense subspaces of X_i for $i = 1, 2$. Then the natural isometry $T_2 \circ T_1^{-1} : T_1E \to T_2E$ can be extended to an isometry between*

X_1 *and* X_2.

Proof. Let $y \in X_1$. Since T_1E is dense in X_1, there exists a sequence (x_n) in E such that $T_1x_n \to y$. Now consider the sequence (T_2x_n) in X_2. Since T_i's are isometries, we have

$$\|T_2x_n - T_2x_m\| = \|T_2(x_n - x_m)\| = \|x_n - x_m\| = \|T_1x_n - T_1x_m\|,$$

by linearity of T_i.
This shows that (T_2x_n) is a Cauchy sequence in X_2, and hence converges, say $z = \lim_{n\to\infty} T_2x_n$.
Define $T : X_1 \to X_2$ by $Ty = z$.
Note that T is well-defined, since if (u_n) is another sequence in E such that T_1u_n converges to y, then T_2u_n also converges to z. This follows from,

$$\begin{aligned} \|T_2x_n - T_2u_n\| &= \|T_2(x_n - u_n)\| \\ &= \|x_n - u_n\| \\ &= \|T_1(x_n - u_n)\| \\ &= \|T_1x_n - T_1u_n\| \to 0 \text{ as } n \to \infty. \end{aligned}$$

Clearly, T is linear. We claim that T is a an isometry from X_1

onto X_2. Since norm is continuous, we have

$$\|y\| = \lim_{n\to\infty} \|T_1 x_n\| = \lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|T_2 x_n\| = \|Ty\|.$$

This implies that T is an isometry.

Now it remains to prove $\mathrm{Im}T = X_2$. Let $z \in X_2$. Since T_2E is dense in X_2, we can choose a sequence $x_n \in E$ such that $\lim_{n\to\infty} T_2 x_n = z$.
Then as above, $(T_1 x_n)$ is a Cauchy sequence in X_1 and let $y = \lim_{n\to\infty} T_1 x_n$. Then, $Ty = z$. This completes the proof. □

Example 2.3.9. Consider the normed space of all continuous functions $C_p[a, b]$ on an interval $[a, b]$ equipped with the norm $\|f\|_p = (\int_a^b |f(x)|^p dx)^{1/p}$, where $1 \leq p < \infty$. It can be shown that $C_p[a, b]$ is dense in the Banach space $L_p[a, b]$ and it is the completion of $C_p[a, b]$. □

Exercise 2.20. Show that $C_p[0, 1]$ is not a Banach space with L^p-norm, $1 \leq p < \infty$.(Hint: On $[0, 1]$, define a sequence of

functions u_n for $n \geq 3$, by

$$u_n(t) = \begin{cases} 0 & 0 \leq t < \frac{1}{2} - \frac{1}{n} \\ nt - \frac{n}{2} + 1 & \frac{1}{2} - \frac{1}{n} \leq t \leq \frac{1}{2} \\ 1 & \frac{1}{2} < t \leq 1 \end{cases}$$

Show that (u_n) is a Cauchy sequence in $C_p[0,1]$ and $\|u_n - u\|_p \to 0$, where

$$u(t) = \begin{cases} 0 & 0 \leq t < \frac{1}{2} \\ 1 & \frac{1}{2} \leq t \leq 1 \end{cases}$$

but $u \notin C_p[0,1]$.)

Chapter 3

Hilbert Spaces

In this module, we introduce a very important mathematical abstract structure, called ***Hilbert spaces***, which were developed by the eminent German mathematician **David Hilbert** (1862-1943). A Hilbert space is the abstraction of the finite dimensional Euclidean spaces of Geometry. The greatest advantage of a Hilbert space is its underlying concept of orthogonality. In order to explore the properties of Hilbert spaces, first we introduce the notion of inner product in linear spaces, which is a

natural generalization of the concept of dot product in $\mathbb{R}^2$.

3.1 Basic Notions

We begin with the precise definition of inner product.

Definition 3.1.1. Let X be a linear space over $\mathbb{C}$. An ***inner product*** (or a scalar product) on X is a complex-valued function from $X \times X \to \mathbb{C}$ such that for all $x, y, z \in X$ and $a, b \in \mathbb{C}$, we have

- Linearity with respect to the first argument :

$$\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle;$$

- Conjugate-Symmetry:

$$\overline{\langle x, y\rangle} = \langle y, x\rangle;$$

- Non-negativeness:

$$\langle x, x\rangle \geq 0, \text{and } \langle x, x\rangle = 0 \text{ if and only if } \ x = 0.$$

A linear space with an inner product is called an ***inner product space.***

Remark 3.1.2. Conjugate-symmetry of the inner product shows that it is *conjugate-linear* (or *semi-linear*) with respect to the second argument; i.e., $\langle x, ay + bz\rangle = \overline{a}\langle x, y\rangle + \overline{b}\langle x, z\rangle$, for all $x, y, z \in X$ and $a, b \in \mathbb{C}$.

To see this, note that

$$\begin{aligned}\langle x, ay + bz\rangle &= \overline{\langle ay + bz, x\rangle} \\ &= \overline{a\langle y, x\rangle + b\langle z, x\rangle} \\ &= \overline{a}\ \overline{\langle y, x\rangle} + \overline{b}\ \overline{\langle z, x\rangle} \\ &= \overline{a}\ \langle x, y\rangle + \overline{b}\ \langle x, z\rangle.\end{aligned}$$

Theorem 3.1.3. ***Cauchy-Schwartz Inequality***
Let X be a linear space with an inner product $\langle\cdot,\cdot\rangle$. Then for any $x, y \in X$,

$$|\langle x, y\rangle| \leq \langle x, x\rangle^{1/2} \cdot \langle y, y\rangle^{1/2}.$$

Proof. Let $p(x) = \langle x, x\rangle^{1/2}$. Then for any number λ, we have:

$$\begin{aligned}\langle x - \lambda y, x - \lambda y\rangle &= \langle x, x - \lambda y\rangle - \lambda\langle y, x - \lambda y\rangle \\ &= \langle x, x\rangle - \overline{\lambda}\langle x, y\rangle - \lambda\langle y, x\rangle + \lambda\overline{\lambda}\langle y, y\rangle \\ &= p(x)^2 - \overline{\lambda\langle y, x\rangle} - \lambda\langle y, x\rangle + |\lambda|^2 p(y)^2 \\ &= p(x)^2 - 2\ \mathrm{Re}(\lambda\langle y, x\rangle) + |\lambda|^2 p(y)^2\end{aligned}$$

so that

$$p(x)^2 - 2\ \mathrm{Re}(\lambda\langle y, x\rangle) + |\lambda|^2 p(y)^2 \geq 0.$$

If $\langle x, y\rangle = 0$, the inequality is trivial. So, assume that $\langle x, y\rangle \neq 0$. Substitute $\lambda = p(x)^2/\langle y, x\rangle$ in the above inequality, we get

$$0 \leq -p(x)^2 + p(x)^4 p(y)^2/|\langle y, x\rangle|^2 \Rightarrow |\langle x, y\rangle|^2 \leq p(x)^2 \cdot p(y)^2,$$

which implies the desired inequality. □

Remark 3.1.4. Equality holds in Cauchy-Schwartz inequality if and only if the set $\{x, y\}$ is linearly dependent. i.e., $x = \lambda y$ for some number λ.

Now we note that an inner product on a linear space X induces a norm on X in a natural way, namely $p(x) = \|x\| = \langle x, x\rangle^{1/2}$ for $x \in X$. Indeed, $p(\lambda x) = |\lambda|\ p(x)$, and to prove the

triangle inequality, note that

$$\mathrm{Re}\langle x, y\rangle \leq |\langle x, y\rangle| \leq p(x)p(y),$$

by the Cauchy-Schwartz inequality. Therefore,

$$\begin{aligned} p(x+y)^2 &= \langle x+y, x+y\rangle \\ &= p(x)^2 - 2\ \mathrm{Re}\langle x, y\rangle + p(y)^2 \\ &\leq (p(x)+p(y))^2 \end{aligned}$$

so that $p(x+y) \leq p(x)+p(y)$.

Thus every inner product space is a normed space and this norm is called "*hilbertian norm*" which will be denoted hereafter by $\|x\|$ instead of $p(x)$.

If an inner product space X is complete with respect to this norm, then X is said to be a ***Hilbert space.*** We often use H to denote a Hilbert space.

Example 3.1.5. For $x = (a_1, a_2, ..., a_n)$ and $y = (b_1, b_2, ..., b_n)$ in $\mathbb{C}^n$, define $\langle x, y\rangle = \sum_{i=1}^{n} a_i\overline{b_i}$. This defines an inner product on $\mathbb{C}^n$, called the *standard inner product* on $\mathbb{C}^n$. The norm on $\mathbb{C}^n$ induced by this inner product is just the Euclidean norm;

$\|x\| = \langle x, x\rangle^{1/2} = (\sum_{i=1}^{n} |a_i|^2)^{1/2}$. Under this norm, $\mathbb{C}^n$ is complete. (*See* Example 2.3.2). Thus, $\mathbb{C}^n$ with the standard inner product is a Hilbert space. □

Example 3.1.6. For elements $x = (a_i)_{i=1}^{\infty}$ and $y = (b_i)_{i=1}^{\infty}$ in ℓ_2, define $\langle x, y\rangle = \sum_{i=1}^{\infty} a_i\overline{b_i}$. This is well defined, since

$$|\sum_{i=1}^{\infty} a_i\overline{b_i}| \leq \sqrt{\sum_{i=1}^{\infty} |a_i|^2} \cdot \sqrt{\sum_{i=1}^{\infty} |b_i|^2} < \infty$$

using Hölder's inequality (*See* Theorem 2.2.5.) One can easily verify that this defines an inner product on ℓ_2. The norm induced by this inner product is given by $\|x\| = \langle x, x\rangle^{1/2} = (\sum_{i=1}^{\infty} |a_i|^2)^{1/2}$, which is nothing but $\|\cdot\|_2$ on ℓ_2. From Example 2.3.4, we have seen that ℓ_2 with this norm is complete, hence ℓ_2 with this inner product is a Hilbert space. □

Example 3.1.7. Consider the space of all continuous functions $C[a, b]$ on an interval $[a, b]$. For $f, g \in C[a, b]$, define

$$\langle f, g\rangle = \int_a^b f(t)\overline{g(t)}dt.$$

From Hölder's inequality for functions (*See* Theorem 2.2.7), we see that the above integral converges absolutely and it can be verified that this defines an inner product on $C[a, b]$.

The norm induced by this inner product is

$$\|f\| = \langle f, f\rangle^{1/2} = (\int_a^b |f(x)|^2 dx)^{1/2} = \|f\|_2,$$

and from Example 2.3.9, $C_2[a, b] = (C[a, b], \|\cdot\|_2)$ is not closed in the Banach space $L_2[a, b]$. Hence $C_2[a, b]$ with the above defined inner product is not a Hilbert space. □

Using Cauchy-Schwartz inequality (Theorem 3.1.3), we now show that the inner product is continuous in both the variables; i.e., $\langle x_n, y_n\rangle \to \langle x, y\rangle$ when $x_n \to x$ and $y_n \to y$. Indeed, we have

$$\begin{aligned} |\langle x_n, y_n\rangle - \langle x, y\rangle| &= |\langle x_n, y_n\rangle - \langle x_n, y\rangle + \langle x_n, y\rangle - \langle x, y\rangle| \\ &\leq |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle| \\ &\leq \|x_n\|\|y_n - y\| + \|x_n - x\|\|y\| \\ &\to 0, \text{ since}(\|x_n\|) \text{ is a bounded sequence.} \end{aligned}$$

Also, for any x, y in an inner product space X, we have

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

This is known as the ***parallelogram law,*** and to see this:

$$\begin{aligned} \|x+y\|^2 + \|x-y\|^2 &= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle \\ &= 2(\langle x, x\rangle + \langle y, y\rangle) \\ &= 2(\|x\|^2 + \|y\|^2). \end{aligned}$$

Using inner product, we introduce the notion of ***orthogonality*** in linear spaces.

We say that two vectors x and y are ***orthogonal*** to each other, denoted by $x \perp y$, if $\langle x, y\rangle = 0$.

If $x \perp y$, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$. This is known as the ***Pythagorean theorem***. Indeed, we have

$$\begin{aligned} \|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\ &= \|x\|^2 + \|y\|^2. \end{aligned}$$

Exercise 3.1. Show that for any x, y in an inner product space

X,

$$\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - \|x-iy\|^2).$$

Remark 3.1.8. We noted that every inner product space is a normed space. It can be shown that a normed space is an inner product space if and only if the norm satisfies the parallelogram law. In other words, a norm $\|\cdot\|$ on a linear space X is induced by an inner product $\langle\cdot,\cdot\rangle$ on it if and only if the norm satisfies the parallelogram law. If it is so, then $\langle\cdot,\cdot\rangle$ is given by the expression in the above exercise 3.1. This result is due to **Jordan** and **von Neumann** (1935). In view of this, we see that a Hilbert space is a complete normed space in which the norm satisfies the parallelogram law. In particular, every Hilbert space is a Banach space, and a Banach space is a Hilbert space only if the norm satisfies the parallelogram law. Using this idea, one can infer that among the sequence spaces $(\ell_p, \|\cdot\|_p);\ 1 \leq p \leq \infty$, $(\ell_2, \|\cdot\|_2)$ is the only Hilbert space.

Example 3.1.9. Consider the space $C_2[a,b]$. From Example 3.1.7, we note that $L_2[a,b]$ is the completion of the normed space $C_2[a,b]$. Any element x of $L_2[a,b]$ can be regarded as a function $x(t)$ such that the Lebesgue integral $\int_a^b |x(t)|^2 dt < \infty$. Using

continuity of inner product, we can extend the inner product on $C_2[a,b]$ to the space $L_2[a,b]$ and indeed, for any $x, y \in L_2[a,b]$, we have $\langle x, y\rangle = \int_a^b x(t)\overline{y(t)}dt$. The space $L_2[a,b]$ with this inner product is a very important Hilbert space. □

Exercise 3.2. If $\{e_i\}_1^n$ are pairwise orthogonal and normalized in a Hilbert space H (i.e., $\|e_i\| = 1$ for $1 \le i \le n$), then for any numbers α_i, show that $\|\sum_1^n \alpha_i e_i\| = (\sum_1^n |\alpha_i|^2)^{1/2}$. (Hint: Use Pythagorean theorem.)

3.2 Orthonormal Systems

Recall that two elements x and y in an inner product space are orthogonal (to each other) if $\langle x, y\rangle = 0$. A set of vectors $\{e_i\}_{i\ge1}$ in an inner product space is said to be an ***orthonormal system*** if the vectors $\{e_i\}$ are pair-wise orthogonal and $\|e_i\| = 1$, for all i. More precisely, we have $\langle e_i, e_j\rangle = \delta_{ij}$ where δ_{ij} is the Kronecker's delta function,

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

The set $\{e_1, e_2, ..., e_n\}$,

$$e_i = (0, 0, ..., 0, 1, 0, ...0),$$

where 1 occurs only at i^{th} place, is an orthonormal system in the Hilbert space $\mathbb{C}^n$ with the standard inner product $\langle x, y\rangle = \sum_{i=1}^{n} a_i\overline{b_i}$, for $x = (a_1, a_2, ..., a_n)$ and $y = (b_1, b_2, ..., b_n)$ in $\mathbb{C}^n$. One can easily check that $\langle e_i, e_j\rangle = \delta_{ij}$.

Exercise 3.3. Show that the set of vectors $\{\frac{1}{\sqrt{2\pi}}e^{int}\}_{n=-\infty}^{n=\infty}$ is an orthonormal system in the Hilbert space $L_2[-\pi, \pi]$ with the inner product $\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)\overline{g(t)}dt$.

Exercise 3.4. Show that the set of vectors $\{\frac{1}{\sqrt{2\pi}}, \frac{cos\ nx}{\sqrt{\pi}}, \frac{sin\ nx}{\sqrt{\pi}}$ for $n = 1, 2, ...\}$ is an orthonormal system in $L_2[-\pi, \pi]$.

We now state and prove ***Bessel's inequality***.

Theorem 3.2.1. *For any orthonormal system $\{e_i\}_{i\geq 1} \subset H$, and for every $x \in H$, we have*

$$\sum_{i\geq 1} |\langle x, e_i\rangle|^2 \leq \|x\|^2.$$

Proof. First we assume that the given orthonormal system $\{e_i\}_{i\geq 1}$

is finite. Consider $y_n = \sum_{i=1}^{n} \langle x, e_i \rangle e_i$. Then, by Cauchy-Schwartz inequality,

$$|\langle y_n, x \rangle| \leq \|x\| \cdot \|y_n\|$$

Also,

$$\begin{aligned} \langle y_n, x \rangle &= \langle \sum_{i=1}^{n} \langle x, e_i \rangle e_i, x \rangle \\ &= \sum_{i=1}^{n} \langle x, e_i \rangle \langle e_i, x \rangle \\ &= \sum_{i=1}^{n} \langle x, e_i \rangle \overline{\langle x, e_i \rangle} \\ &= \sum_{i=1}^{n} |\langle x, e_i \rangle|^2 \end{aligned}$$

and,

$$\begin{aligned}
\|y_n\|^2 &= \langle y_n, y_n\rangle \\
&= \langle \sum_{i=1}^{n}\langle x, e_i\rangle e_i, \sum_{j=1}^{n}\langle x, e_j\rangle e_j\rangle \\
&= \sum_{i=1}^{n}\langle x, e_i\rangle\langle e_i, \sum_{j=1}^{n}\langle x, e_j\rangle e_j\rangle \\
&= \sum_{i=1}^{n}\langle x, e_i\rangle \sum_{j=1}^{n}\overline{\langle x, e_j\rangle}\langle e_i, e_j\rangle \\
&= \sum_{i=1}^{n}\langle x, e_i\rangle\overline{\langle x, e_i\rangle} \\
&= \sum_{i=1}^{n}|\langle x, e_i\rangle|^2
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{i=1}^{n}|\langle x, e_i\rangle|^2 &= |\langle y_n, x\rangle| \\
&\leq \|x\|\cdot\|y_n\| \\
&= \|x\|\cdot(\sum_{i=1}^{n}|\langle x, e_i\rangle|^2)^{1/2}
\end{aligned}$$

which proves the desired inequality for a finite set $\{e_i\}_1^n$.

If the given orthonormal system $\{e_i\}_{i\geq 1}$ is infinite, then let $n \to \infty$ in $\sum_{i=1}^{n} |\langle x, e_i\rangle|^2 \leq \|x\|^2$, we get $\sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2 \leq \|x\|^2$. □

Corollary 3.2.2. *For any $x \in H$ and any orthonormal system $\{e_i\}_1^\infty$, there exists $y \in H$ such that $y = \sum_{i=1}^{\infty} \langle x, e_i\rangle e_i$.*

Proof. From the infinite case of the equality in Exercise 3.2, and using Bessel's inequality (Theorem 3.2.1), we note that $y_n = \sum_{i=1}^{n} \langle x, e_i\rangle e_i$ is a Cauchy sequence in the Hilbert space H. Due to the completeness of H, the series $y_n = \sum_{i=1}^{n} \langle x, e_i\rangle e_i$ converges to some element $y \in H$. Thus, there exists $y \in H$ such that $y = \sum_{i=1}^{\infty} \langle x, e_i\rangle e_i$.

□

A system of vectors $\{x_i\}_{i\geq 1}$ in a normed space X is called a **complete system** if the linear span

$$\{\sum_{i=1}^{n} \alpha_i x_i : \text{ for all scalars } \alpha_i, \text{for all } n \in \mathbb{N}\}$$

of the vectors $\{x_i\}_{i\geq 1}$ is a dense set in X.

Example 3.2.3. From the Weierstrass approximation theorem in real analysis, the set of polynomials is dense in the space $C[0,1]$ so that the system $\{t^n\}_{n\geq 0}$ is complete in the space $C[0,1]$. Since $C[0,1]$ is dense in $L_2[0,1]$, it follows that the system $\{t^n\}_{n\geq 0}$ is complete in the space $L_2[0,1]$ too.

Also, from the Weierstrass theorem, the set of trigonometric polynomials is dense in the space $C[-\pi,\pi]$. Hence it follows that the sets $\{1, cos\ nx, sin\ nx : n = 1,2,3,...\}$ and $\{e^{int}\}_{n=-\infty}^{\infty}$ are complete systems in $L_2[-\pi,\pi]$.

Lemma 3.2.4. *If $\{f_i\}$ is a complete system in a Hilbert space H and $x \perp f_i$ for all i, then $x = 0$.*

Proof. Since $\{f_i\}$ is a complete system in H, linear span of $\{f_i\}$ is dense in H and since $x \perp f_i$ for all i, we have $x \perp$ span$\{f_i\}$ for all $x \in H$. Hence for any $x \in H$, there exists a sequence x_n converging to x such that $x_n \perp x$ for all $n \in \mathbb{N}$, so that $\|x\|^2 = \langle x, x\rangle = \lim\langle x_n, x\rangle = 0$, implies $x = 0$. □

Exercise 3.5. If $\{f_i\}$ is a system of vectors in a Hilbert space H, with the property that $x \perp f_i$ for all i, implies $x = 0$, then show that $\{f_i\}$ is a complete system.

3.3 Orthogonal Bases

Given a linearly independent set of vectors $\{x_i\}$ in an inner product space, there is an inductive procedure, called ***Gram - Schmidt orthonormalization process***, to construct a set of orthonormal vectors $\{e_i\}$ such that $\text{span}\{x_i\}_1^n = \text{span}\{e_i\}_1^n$ for every $n = 1, 2, 3, ...$ We set $e_1 = x_1/\|x_1\|$ and define inductively for $n = 2, 3, ...,$

$e_n = (x_n - y_n)/\|x_n - y_n\|$ where $y_n = \sum_{i=1}^{n-1} \langle x_n, e_i \rangle e_i$.
This definition makes sense since we always have $\|x_n - y_n\| \neq 0$, due to the linear independence of $\{x_i\}$. (Verify it!)
This sequence $\{e_i\}_1^\infty$ is an orthonormal system, i.e., $\langle e_i, e_j \rangle = \delta_{ij}$, such that $\text{span}\{x_i\}_1^n = \text{span}\{e_i\}_1^n$ for every $n = 1, 2, 3, ...$

We know that the set of rational numbers is a countable dense subset of the set of real numbers. We now introduce a special class of normed spaces with such a property as follows.

Definition 3.3.1. A normed space X is called a **separable space** if there exists a dense countable set in X.

We now present a characterization of separable Hilbert spaces.

Theorem 3.3.2. *A Hilbert space H is separable if and only if there exists a complete orthonormal system $\{e_i\}_{i\geq 1}$ in H.*

Proof. First, assume that H is separable. Then by definition, there exists a countable subset $Y = \{y_i\}_{i\geq 1}$, which is dense in H. We now inductively construct a linearly independent subset $\{x_i\}_{i\geq 1}$ of Y such that for every n, $\text{span}\{x_i\}_1^n = \text{span}\{y_i\}_1^{N_n}$ for some $N_n \in \mathbb{N}$. For this, we choose x_1 as the first non-zero element of $Y = \{y_i\}_{i\geq 1}$, and then inductively, if $\{x_i\}_1^n$ are chosen, we take x_{n+1} as the first y_i, $i > N_n$, which is linearly independent of $\{x_i\}_1^n$. Clearly, $\text{span}\{x_i\}_{i\geq 1}$ is also dense in H. If H is of finite dimension, say $\dim H = n$, then this procedure will stop after a finite number of steps, producing a basis $\{x_i\}_1^n$. If $\dim H$ is infinite, this process will continue indefinitely. In any case, by applying Gram-Schmidt orthogonalization procedure to the system $\{x_i\}_{i\geq 1}$, we find a complete orthonormal system $\{e_i\}_{i\geq 1}$ in H. This completes one direction of the proof.

For the other direction, if $\{e_i\}_{i\geq 1}$ is complete in H, then the set of all finite sums $\sum \alpha_i e_i$ with rational coefficients α_i forms a countable dense subset of H, and hence H is separable. □

Recall the definition of basis for a vector space. We now

define the concept of a basis for a normed space.

Definition 3.3.3. A sequence $\{x_i\}_{i=1}^{\infty}$ is called ***a basis of a normed space*** X if for every $x \in X$, there exists a unique series $\sum_{i\geq 1} a_i x_i$ that converges to x.

Note the difference; our usual notion of basis requires that any element of the space can be expressed as a unique finite linear combination of basis vectors; but here we allow infinite sums too.

Next, we prove that any complete orthonormal system in a Hilbert space is a basis for it.

Theorem 3.3.4. *A complete orthonormal system* $\{e_i\}_{i=1}^{\infty}$ *in a Hilbert space* H *is a basis in* H.

Proof. Let $\{e_i\}_{i=1}^{\infty}$ be a complete orthonormal system in a Hilbert space H and $x \in H$. By the Bessel's inequality (Theorem 3.2.1), $\sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2 < \infty$. Hence by Corollary 3.2.2, the element $y = \sum_{i=1}^{\infty} \langle x, e_i\rangle e_i$ exists in H.
Also, it is easy to check that $\langle y - x, e_i\rangle = 0$ for all i, implies that $y - x \perp e_i$ for every i. Therefore, by Lemma 3.2.4, we get $y - x = 0$ or $y = x$. Thus, $x = \sum_{i=1}^{\infty} \langle x, e_i\rangle e_i$.

Now let $x = \sum_{i=1}^{\infty} a_i e_i$, then $\langle x, e_i \rangle = a_i$, showing that the expression $x = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i$ is unique. □

Combining the results in Theorem 3.3.2 and Theorem 3.3.4 we get the following corollary.

Corollary 3.3.5. *Every separable Hilbert space has an orthonormal basis.*

One can easily check that the system of vectors described in exercises 3.3 and 3.4 are orthonormal bases for the Hilbert space $L_2[-\pi, \pi]$.

Theorem 3.3.6. *Let $\{e_i\}_{i \geq 1}$ be an orthonormal system in a Hilbert space H. Then $\{e_i\}_{i \geq 1}$ is a basis in H if and only if for all $x \in H$,*

$$\|x\|^2 = \sum_{i \geq 1} |\langle x, e_i \rangle|^2.$$

This identity is known as ***Parseval's identity***

Proof. Suppose the orthonormal system $\{e_i\}_{i \geq 1}$ is a basis in H.

Then for any $x \in H$, we have $x = \sum_{i\geq 1}\langle x, e_i\rangle e_i$, so that

$$\begin{aligned}\|x\|^2 &= \langle x, x\rangle \\ &= \langle \sum_{i\geq 1}\langle x, e_i\rangle e_i, \sum_{j\geq 1}\langle x, e_j\rangle e_j\rangle \\ &= \sum_{i\geq 1}|\langle x, e_i\rangle|^2.\end{aligned}$$

To prove the converse, note that
$\|x - \sum_1^n\langle x, e_i\rangle e_i\|^2 = \|x\|^2 - \sum_1^n|\langle x, e_i\rangle\|^2 \to 0$ as $n \to \infty$, since by assumption, we have $\|x\|^2 = \sum_{i\geq 1}|\langle x, e_i\rangle|^2$.

Therefore, $x = \sum_{i\geq 1}\langle x, e_i\rangle e_i$ for any $x \in H$, showing that $\{e_i\}_{i\geq 1}$ is a basis in H. □

Remark 3.3.7. If the Parseval's identity holds for a particular x in H, then we have $x = \sum_{i\geq 1}\langle x, e_i\rangle e_i$. Thus, if the Parseval's identity holds for every member of a dense subset of H, we see that $\{e_i\}_{i\geq 1}$ is a complete system and hence by Theorem 3.3.4, it is a basis in H.

Next we show that any two separable infinite dimensional Hilbert spaces are the same in the following sense.

Theorem 3.3.8. *Any two separable infinite dimensional Hilbert spaces H_1 and H_2 are isometrically equivalent; i.e., there exists a linear isomorphism onto $T : H_1 \to H_2$ such that $\|Tx\| = \|x\|$ for every $x \in H_1$ and, moreover, $\langle Tx, Ty\rangle_{H_2} = \langle x, y\rangle_{H_1}$ for every x and y in H_1.*

Proof. Let H be a separable infinite dimensional Hilbert space. To prove the theorem, it is enough to show that H is isometrically equivalent to ℓ_2, the space of all square summable scalar sequences, which is also a separable infinite dimensional Hilbert space.

Let $\{f_i\}_1^\infty$ be an orthonormal basis for H. Then for every $x \in H$, $x = \sum_{i\geq 1}\langle x, f_i\rangle f_i$ and $\|x\|^2 = \sum_{i\geq 1}|\langle x, f_i\rangle|^2$.

Let $\{e_i\}_1^\infty$ be the natural basis of ℓ_2. Define $T : H \to \ell_2$ by $Tx = \sum_1^\infty \langle x, f_i\rangle e_i = (\langle x, f_1\rangle, \langle x, f_2\rangle, \langle x, f_3\rangle, ...) \in \ell_2$, using

Bessel's inequality. Then T is clearly linear and

$$\begin{aligned}\|Tx\|^2 &= \langle Tx, Tx\rangle \\ &= \langle \sum_1^\infty \langle x, f_i\rangle e_i, \sum_1^\infty \langle x, f_j\rangle e_j\rangle \\ &= \sum_1^\infty |\langle x, f_i\rangle|^2 \\ &= \|x\|^2,\end{aligned}$$

showing that $\|Tx\| = \|x\|$ for every $x \in H$.

Now let $(a_1, a_2, a_3, ...) \in \ell_2$. Then $\sum_1^\infty a_i f_i$ converges in H (Verify that its partial sums forms a Cauchy sequence in H and hence converges.) and let $x = \sum_1^\infty a_i f_i \in H$. Then $\langle x, f_i\rangle = a_i$ for all i and hence $Tx = \sum_1^\infty \langle x, f_i\rangle e_i = (a_1, a_2, a_3, ...)$. This shows that T is onto. Also,

$$\begin{aligned}\langle Tx, Ty\rangle_{\ell_2} &= \langle \sum_1^\infty \langle x, f_i\rangle e_i, \sum_1^\infty \langle y, f_j\rangle e_j\rangle \\ &= \sum_1^\infty \langle x, f_i\rangle\langle y, f_i\rangle \\ &= \langle x, y\rangle_H\end{aligned}$$

This completes the proof. □

Exercise 3.6. Let $\{\varphi_i(t)\}_1^\infty$ be a sequence of continuous functions that forms an orthnormal basis for $L_2[a,b]$. Then show that the system

$$\{\varphi_i(t)\varphi_j(\tau) = \psi_{ij}(t,\tau)\}_{i,j=1}^\infty$$

is an orthonormal basis of $L_2([a,b]\times[a,b])$, which is the completion of the space of square integrable (continuous) functions of two variables with norm $\|f(t,\tau)\| = \sqrt{\int_a^b \int_a^b |f(t,\tau)|^2 dt d\tau}$.

3.4 Orthogonal Decomposition

We begin with the definition of projections. Let L be a closed subspace of a Hilbert space H. We define the distance of any $x \in H$ to L as $\rho(x,L) = \inf_{y\in L} \|x-y\|$. If this infimum is achieved at some $y \in L$, we define $y = P_L x$, and call this y as the ***projection*** of x onto L.

We now consider a particular case that the subspace L considered is separable. Let $\{e_i\}_{i\geq 1}$ be an orthonormal basis in L and let $x \in H$. Set $y = \sum_{i\geq 1}\langle x, e_i\rangle e_i$. Then $y \in L$ and clearly

$x - y \perp e_i$ for all i, which implies $x - y \perp L$.

Therefore, for any $z \in L$, we have $x - z = (x - y) + (y - z)$ and $\|x - z\|^2 = \|x - y\|^2 + \|y - z\|^2$, since $x - y \perp y - z$. It follows

$$\inf\{\|x - z\| \mid z \in L\} = \|x - y\|$$

and $y = P_L x$. Also, such a y is unique.

Exercise 3.7. Show that a closed subspace of a separable Hilbert space is separable.

Before proving the existence of a unique projection in the non-separable case, we prove the following more general fact.

Theorem 3.4.1. *Let M be a convex closed set in a Hilbert space H. Then there exists a unique $y \in M$ such that*

$$\rho(x, M) = \|x - y\|,$$

where $\rho(x, M) = \inf_{w \in M} \|x - w\|$, the distance of x to the set M.

Proof. Let $d = \rho(x, M) = \inf_{w \in M} \|x - w\|$. As d is an infimum, we can choose a sequence $y_n \in M$ and $\|x - y_n\| \to d$. We claim

that this sequence $\{y_n\}$ is a Cauchy sequence. From parallelogram law, we find that

$$2(\|x-y_n\|^2+\|x-y_m\|^2)=\|2x-(y_n+y_m)\|^2+\|y_n-y_m\|^2.$$

By the convexity of M, $(y_n+y_m)/2\in M$ and hence

$$\|x-(y_n+y_m)/2\|\geq d,$$

we see that the right side of the above equality tends to $4d^2$ as $n,m\to\infty$.
But the right side is not smaller than $4d^2+\overline{lim}\|y_n-y_m\|^2$.
From this, we conclude that $\|y_n-y_m\|\to 0$ as $n,m\to\infty$ and hence $\{y_n\}$ is a Cauchy sequence in M so that there exists $y\in H$ such that $y=\lim_{n\to\infty}y_n$. As M is a closed subspace of H, we see that $y\in M$ and $\rho(x,M)=\|x-y\|$.

Suppose $\rho(x,M)=\|x-y_1\|=\|x-y_2\|$ for $y_1,y_2\in M$, then again from parallelogram law and the convexity of M, it follows that $y_1=y_2$. This proves the uniqueness of y. □

Instead of closed convex set M, if we take a closed subspace L in the above theorem, we see that there is a unique projection $y=P_Lx$. Also, note that $\rho(x,L)=\|x-y\|$ for some $y\in L$ if

and only if $x - y \perp L$. To see this, first let $y \in L$ be such that $x - y \perp L$. Then, for any $z \in L$

$$\|x - z\|^2 = \|x - y\|^2 + \|y - z\|^2 \geq \|x - y\|^2,$$

shows that y gives the distance and it is unique.

To see the opposite direction, assume that $y \in L$ is the projection $P_L x$. Then for any $z \in L$,

$$\|x-y\|^2 \leq \|x-(y+\lambda z)\|^2 = \|x-y\|^2-2Re\lambda\langle z, x-y\rangle+|\lambda|^2\|z\|^2.$$

This implies,

$$2Re\lambda\langle z, x - y\rangle \leq |\lambda|^2\|z\|^2.$$

Take $\lambda = t\overline{\langle z, x - y\rangle}$ for $t \in \mathbb{R}$. Then

$$2t|\langle z, x - y\rangle|^2 \leq t^2|\langle z, x - y\rangle|^2\|z\|^2$$

for all $t \in \mathbb{R}$. Letting $t \to 0$, we get $\langle z, x - y\rangle = 0$ and hence every $z \in L$ is orthogonal to $x - y$. Thus, we have proved the following theorem.

Theorem 3.4.2. *For all $x \in H$, there exists a unique $y \in L$ such that $x - y \perp L$ and $y = P_L x$. Then, $x = (x - y) + y$ and*

$\|x\|^2 = \|x - y\|^2 + \|y\|^2$. □

For any subspace L of a Hilbert space H, we define the ***orthogonal complement***, denoted by $L^\perp$ of L, as :

$$L^\perp = \{x \in H \ : \ x \perp L\} = \{x \in H \ : \ x \perp y, \text{for every } y \in L\}.$$

Exercise 3.8. For any subspace L of H, prove that $L^\perp$ is a closed subspace of H. Also, prove that $L \cap L^\perp = \{0\}$.

The following theorem is known as *the Projection Theorem.*

Theorem 3.4.3. *For every closed subspace L of a Hilbert space H,*

$$L \oplus L^\perp = H.$$

Moreover,$(L^\perp)^\perp = L$.

Proof. Let $x \in H$. For any closed subspace $L \hookrightarrow H$, by Theorem 3.4.2, there exists a unique $y \in L$ such that $x = (x-y)+y$ with $x - y \in L^\perp$. (i.e., $y = P_L x$.).

Also, from Exercise 3.8, $L \cap L^\perp = \{0\}$. If $x = z_1 + y_1$ with $z_1 \in L^\perp$ and $y_1 \in L$, then $z_1 + y_1 = (x - y) + y$, so that $z_1 - (x - y) = y - y_1$.

But L and $L^\perp$ are subspaces and $L \cap L^\perp = \{0\}$, so that $z_1 = (x - y)$ and $y_1 = y$. This shows that $L \oplus L^\perp = H$.

Now we show that $(L^\perp)^\perp = L$.

Let $y \in L$. If $x \in L^\perp$, then $\langle y, x\rangle = 0 \ \Rightarrow y \in (L^\perp)^\perp$. This implies that $L \subset (L^\perp)^\perp$.

Next, let $y \in (L^\perp)^\perp$. Since $L \oplus L^\perp = H$, we have $y = z_1 + z_2$ with $z_1 \in L$ and $z_2 \in L^\perp$. Since $L \subset (L^\perp)^\perp$, $z_1 \in (L^\perp)^\perp$ and since $(L^\perp)^\perp$ is a subspace (*See* Exercise 3.8), we see that $y - z_1 \in (L^\perp)^\perp \implies z_2 \in (L^\perp)^\perp$ But $z_2 \in L^\perp$ and hence $z_2 \in L^\perp \bigcap (L^\perp)^\perp = \{0\}$, implies $z_2 = 0$.
Hence $y = z_1 \in L \ \Rightarrow (L^\perp)^\perp \subset L$.

Thus, we have $(L^\perp)^\perp = L$. □

Exercise 3.9. Let L_1 and L_2 be closed subspaces of a Hilbert space H such that $L_1 \hookrightarrow L_2 \hookrightarrow H$. Let $x_2 = P_{L_2}x$ for any $x \in H$. Prove that $P_{L_1}x = P_{L_1}x_2$.

Lemma 3.4.4. *If E is a closed subspace of a Hilbert space H, and codim$E = 1$, then the subspace $E^\perp$ is 1-dimensional.*

Proof. By Theorem 3.4.3, we have $E \oplus E^\perp = H$. Suppose there exists two linearly independent vectors x_1, x_2 in $E^\perp$. Then by

Gram-Schmidt orthogonalization process, we get two orthogonal vectors y_1, y_2 in $E^\perp$. Now, if $\alpha y_1 + \beta y_2 = z \in E$, since $y_1, y_2 \in E^\perp$, it follows that $\alpha = \beta = 0$ and hence $z = 0$. This shows that $y_1 + E,\ y_2 + E$ are linearly independent vectors in H/E, which is a contradiction to the assumption that $\text{codim}E = 1$. This shows that the subspace $E^\perp$ is 1-dimensional.

□

3.5 Linear Functionals

Linear functionals on a general linear space E over a field $K = \mathbb{R}$ or $\mathbb{C}$, are just linear transformations from E to K. More precisely, we have the following definition.

Definition 3.5.1. Let E be a linear space over $\mathbb{R}$ (or $\mathbb{C}$). Linear functionals are functions $f : E \mapsto \mathbb{R}$ (or correspondingly $\mathbb{C}$) such that

$$f(\lambda x + \mu y) = \lambda f(x) + \mu f(y),$$

for all $x, y \in E$ and for all $\lambda, \mu \in \mathbb{R}$ (*or* $\mathbb{C}$).

For any linear functional f on E, the *kernel* of f,

$\ker f = \{x \in E \ : \ f(x) = 0\}$ is a linear subspace of E, which we often denote by H_f.

Example 3.5.2. Consider the space c_0 of all sequences that converges to zero. For any scalar sequence (b_i) with $\sum |b_i| < \infty$, i.e., $(b_i) \in \ell_1$, define the linear functional on c_0 as $f(x) = \sum_1^\infty a_i b_i$. (Verify it!) We will identify such a functional f with the element (b_i) of ℓ_1.

In a similar way, in the linear space ℓ_p, $1 < p < \infty$, with any $(b_i) \in \ell_q$, one can define a linear functional f by $f(x) = \sum_1^\infty a_i b_i$. By using Hölder's inequality for sequences (*See* Theorem 2.2.5),

$$|f(x)| \leq \sum_1^\infty |a_i| \cdot |b_i| \leq \|x\|_{\ell_p} \|f\|_{\ell_q} < \infty,$$

we see that f is defined for any $x = (a_i) \in \ell_p$. We identify this functional with the element (b_i) of ℓ_q.

Example 3.5.3. Now we will give two linear functionals in the space $C[0, 1]$.
Define $F(x) = \int_0^1 x(t) f(t) dt$, where f is some integrable function. This is well defined since $|F(x)| \leq \max_t |x(t)| \int_0^1 |f(t)| dt$.

Evaluation at some fixed point $\alpha \in [0, 1]$, defined by $\delta_\alpha(x) =$

$x(\alpha)$ is a linear functional on $C[0,1]$, since $|\delta_\alpha(x)| \leq \|x\|_{C[0,1]}$.

For any linear space E, the set of all linear functionals on E is a linear space and is denoted by $E^{\#}$. The proof of the following theorem is left as an exercise.

Theorem 3.5.4. *Let f be a non-zero linear functional on a linear space E. Then,*

- *codim ker $f = 1$*
- *If $f, g \in E^{\#}\backslash\{0\}$ and ker f = ker g, then there exists $\lambda \neq 0$ such that $\lambda f = g$.*
- *Let $L \hookrightarrow E$ and codim $L = 1$. Then there exists $f \in E^{\#}$ such that ker $f = L$.*

3.6 Bounded Linear Functionals

Let $X = (E, \|\cdot\|)$ be a normed space. A linear functional on X is said to be a *bounded functional* if there exists $C > 0$ such that $|f(x)| \leq C\|x\|$ for all $x \in X$. i.e., if f is bounded on bounded sets.

We say that a linear functional f on X is a continuous functional if $x_n \to x$, implies that $f(x_n) \to f(x)$.

Let X^* denotes the set of all bounded linear functionals on the normed space X. Then X^* is clearly a linear space with respect to point-wise addition and scalar multiplication. Define a norm in X^* as follows: For $f \in X^*$, let

$$\|f\|^* = \sup_{x \neq 0} |f(x)|/\|x\| = \sup_{\|x\|=1} |f(x)|$$

(Verify that this indeed defines a norm on X^*.) In particular, we see that

$$|f(x)| \leq \|f\|^* \cdot \|x\|.$$

We often write $\|f\|$ instead of $\|f\|^*$.

If f is a bounded linear functional on X, then for any sequence (x_n) with $x_n \to x_0$, we have

$$|f(x_n) - f(x_0)| = |f(x_n - x_0)| \leq \|f\| \cdot \|x_n - x_0\| \to 0,$$

as $n \to \infty$. This shows that f is a continuous linear functional.

Now assume that f is a continuous linear functional. Suppose f is not bounded. Then for any $n \in \mathbb{N}$, there exists x_n

in X, with $\|x_n\| = 1$ and $|f(x_n)| > n$. Consider the sequence $y_n = x_n/n$. Then, $y_n \to 0$, so that we must have $f(y_n) \to 0$. But $|f(y_n)| = |f(x_n)|/n \geq 1$, which is a contradiction. Thus, we have observed that:

A linear functional f on a normed space is a bounded functional if and only if f is a continuous functional.

The linearity of f is very crucial and we can observe that ***if a linear functional f on X, is continuous at*** 0, ***then it is continuous at any*** $x \in X$. To see this, let $x_n \to x$, then $x_n - x \to 0$, and by the continuity at 0, we have $f(x_n) - f(x) = f(x_n - x) \to f(0) = 0$, implies that $f(x_n) \to f(x)$.

Exercise 3.10. A linear functional f on a normed space X is continuous if and only if $\ker f$ is a closed subspace of X.

Exercise 3.11. For any bounded linear functional f on X, prove that

$$\begin{aligned} \|f\| &= \sup\{|f(x)| \ : \ \|x\| \leq 1\} \\ &= \sup\{1/\|x\| \ : \ f(x) = 1\} \\ &= \frac{1}{\inf\{\|x\| \ : \ f(x) = 1\}} \end{aligned}$$

Now, we turn our attention to bounded linear functionals on a Hilbert space. For any fixed element y in a Hilbert space H, define $f(x) = \langle x, y\rangle$. Clearly, f is a linear functional in H and by using Cauchy-Schwartz inequality (Theorem 3.1.3), we see that $|f(x)| = |\langle x, y\rangle| \leq \|x\| \cdot \|y\|$.

Now suppose that f is any linear functional on H that is continuous too. Does there exist a $y \in H$ such that $f(x) = \langle x, y\rangle$?. The answer is yes, and this is the content of the following result, known as ***Riesz representation theorem.***

Theorem 3.6.1. *For every $\varphi \in H^*$, there exists a unique $y \in H$ such that*

$$\varphi(x) = \langle x, y\rangle, \ x \in H;$$

(i.e., any continuous linear functional on a Hilbert space H is represented by a unique element of the same space H.) Moreover, this correspondence is an isometry: $\|\varphi\|^* = \|y\|$. Conversely, for any $y \in H$, the expression $\varphi(x) = \langle x, y\rangle$ defines a continuous linear functional on H, with the norm $\|\varphi\|^* = \|y\|$.

Proof. For any fixed $y \in H$, define $\varphi_y(x) = \langle x, y\rangle$.
From above discussions, we see that φ_y is linear and satisfies $|\varphi_y(x)| \leq \|x\| \cdot \|y\|$. This implies $\|\varphi_y\|^* \leq \|y\|$.

Setting $x = y$, we get $\varphi_y(y) = \langle y, y\rangle = \|y\|^2 \leq \|\varphi_y\|^*\|y\|$, implies $\|y\| \leq \|\varphi_y\|^*$.
Thus, $\|\varphi_y\|^* = \|y\|$.

Now let us show that any continuous linear functional φ on H is of the form $\varphi(x) = \langle x, y\rangle$, for some $y \in H$.
If $\varphi = 0$, we can simply take $y = 0$.
If $\varphi \neq 0$, then $L = \ker\varphi$ is closed in H and $\text{codim}L = 1$. Therefore, $L^\perp$ is 1-dimensional, so that $L^\perp = \text{span}\{\hat{y}\}$ for some $\hat{y} \in H\backslash\{0\}$.

Using this vector $\hat{y}$, define the linear functional $\varphi_{\hat{y}} = \langle x, y\rangle$. Then $\ker\varphi_{\hat{y}} = \{\hat{y}\}^\perp = (L^\perp)^\perp = L$. Thus, $\ker\varphi_{\hat{y}} = \ker\varphi$ and hence by the second part of Theorem 3.5.4, we get $\varphi = \lambda\varphi_{\hat{y}}$, for some scalar λ.
Put $y = \overline{\lambda}\hat{y}$. Then, $\varphi(x) = \lambda\varphi_{\hat{y}}(x) = \lambda\langle x, \hat{y}\rangle = \langle x, y\rangle$. Also, we have noted in this case, $\|\varphi\| = \|y\|$.

Now to prove the uniqueness of y, let $\varphi(x) = \langle x, y\rangle = \langle x, y_1\rangle$ for all $x \in H$. Then, let $x = y - y_1$, we get $\langle y - y_1, y\rangle = \varphi(y - y_1) = \langle y - y_1, y_1\rangle$ implies $\langle y - y_1, y - y_1\rangle = 0$, which shows that $y = y_1$. This completes the proof. □

Most of the results about Hilbert spaces that we discussed

deal with separable Hilbert spaces, but the Projection theorem (Theorem 3.4.3) and the Riesz representation theorem (Theorem 3.6.1) are valid without such assumption. We now give an example for a non-separable Hilbert space.

Consider the linear space L of continuous functions on $\mathbb{R}$ such that

$$p(f) = (\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(x)|^2 dx)^{1/2}$$

is defined for $f \in L$. This p is only a semi norm on L, since it may happen that $p(f) = 0$ for some non-zero functions too. (For instance, $p(f)$ is zero for any function with finite support.)

Let $N = \ker p$ and consider the quotient space L/N. Then it is a normed space and p is a norm on it. Let H be the completion of this space with respect to the norm p and the inner product

$$\langle f, g\rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\overline{g(x)}dx.$$

Note that the set $\{e^{i\lambda t}\}_{\lambda\in\mathbb{R}}$ is an uncountable orthonormal system in H, and hence H is not separable. (If a Hilbert space H has an uncountable orthonormal system, say $\{e_t\}_{t\in\mathbb{R}}$, then the open balls $\{B(e_t, 1/2)\}$ are pair wise disjoint and hence H can-

not have a countable dense subset, as any such set should put at least one element in each of the above uncountable, disjoint balls.)

Exercise 3.12. Prove that for any two subspaces L_1 and L_2 of a Hilbert space, $(L_1 + L_2)^{\perp} = (L_1^{\perp} \bigcap L_2^{\perp})$.

Exercise 3.13. Let L_1 and L_2 be closed subspaces of a Hilbert space H. Then prove $(L_1 \bigcap L_2)^{\perp}$ equals the closure of $L_1^{\perp} + L_2^{\perp}$.

Exercise 3.14. If H_1 and H_2 are two Hilbert spaces and if $T : H_1 \to H_2$ is an isometry, then prove that $\langle Tx, Ty \rangle = \langle x, y \rangle$ for every $x, y \in H_1$.

Chapter 4

The Dual Space of a Normed Space

This chapter deals with the duality between a normed space X and the space X^* of all bounded linear functionals on it. The linear space X^* of all bounded linear functionals on X is equipped with the norm,

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|},$$

we call this norm as the dual norm (i.e., dual to the original norm of X), and the normed space X^* as the dual space (i.e., dual to the space X). From the definition of dual norm, it is clear that

$$|f(x)| \leq \|f\| \cdot \|x\|.$$

It can be proved that ***the dual space*** X^* ***is always complete and hence is a Banach space for any normed space*** X.

4.1 The Hahn-Banach Theorem and its Consequences.

We now state ***Hahn-Banach extension theorem***, which is one of the fundamental theorem in the theory of normed spaces.

Theorem 4.1.1. *(**Hahn-Banach**)*
Let X be a normed space over $\mathbb{R}$, E be a subspace of X, and $f_0 \in E^*$. Then, there exists an extension $f \in X^*$ such that $f|_E = f_0$ (i.e., $f(x) = f_0(x)$ for all $x \in E$) and $\|f\|_{X^*} = \|f_0\|_{E^*}$. i.e.,

$$\sup_{x \in X \setminus \{0\}} \frac{|f(x)|}{\|x\|} = \sup_{x \in E \setminus \{0\}} \frac{|f_0(x)|}{\|x\|}.$$

We now derive some corollaries to this theorem.

Corollary 4.1.2. *For all $x_0 \in S(X) = \{x \in X \ : \ \|x\| = 1\}$, the unit sphere of X, there exists $f_0 \in X^*$ such that $\|f_0\|_{X^*} = 1$, and $f_0(x_0) = 1$. So the functional f_0 attains its supremum on the unit ball on the vector x_0.*

Proof. Let $E_0 = \{\lambda x_0 \ : \ \lambda \in \mathbb{R}$. Then, E_0 is a 1-dimensional subspace of X. Define the functional φ on E_0 as $\varphi(\lambda x_0) = \lambda$. Then $\|\varphi\|_{E_0^*} = 1$. By the Hahn-Banach theorem, there exists an extension f_0 with the desired properties. □

Corollary 4.1.3. *For all $x_0 \in X$, there exists $f_0 \in X^* \backslash \{0\}$ such that $f_0(x_0) = \|f_0\| \cdot \|x_0\|$.*

Proof. If $x_0 = 0$, then the desired property holds for any $f_0 \in X^* \backslash \{0\}$. If $x_0 \neq 0$, set $y = x_0/\|x_0\|$. Then $y \in S(X)$ and by above corollary 4.1.2, there exists $f_0 \in X^* \backslash \{0\}$ with the desired properties. □

Corollary 4.1.4. *For all $x_1 \in X$ and for all $x_2 \in X$ such that $x_1 \neq x_2$, there exists $f \in X^*$ satisfying $f(x_1) \neq f(x_2)$.*

Proof. Apply Corollary 4.1.3 for the element $x_0 = x_1 - x_2$, we get $f_0 \in X^* \backslash \{0\}$ such that $f_0(x_0) = f_0(x_1 - x_2) = f_0(x_1) - f_0(x_2) = \|f_0\|\|x_1 - x_2\| \neq 0$, implies $f_0(x_1) \neq f_0(x_2)$. □

Corollary 4.1.5. *X^* is a total set. i.e., if $f(x) = 0$ for all $f \in X^*$, then $x = 0$.*

Proof. If $x \neq 0$, then by Corollary 4.1.4, there exists $f \in X^*$ satisfying $f(x) \neq 0 = f(0)$, which is a contradiction. □

Corollary 4.1.6. *Let X be a normed space over $\mathbb{R}$ and let $L \hookrightarrow X$ be a subspace. If $x \in X$ is such that dist$(x, L) = d > 0$, then there exists $f \in X^*$ such that $\|f\| = 1$, $f(L) = 0$ and $f(x) = d$.*

Proof. Let $L_1 = \text{span}\{x, L\}$, i.e.,

$$L_1 = \{\lambda x + y \ : \lambda \in \mathbb{R}, y \in L\}.$$

Then L_1 is a subspace of X, and define

$f_0 : L_1 \to \mathbb{R}$ by $f_0(\lambda x + y) = \lambda \cdot d$.

Then f_0 is well-defined since any element $z \in L_1$ has a unique expression of the form $z = \lambda x + y$ with $\lambda \in \mathbb{R}$ and $y \in L$. Clearly f_0 is linear, hence f_0 is a linear functional on L_1, $f_0(L) = 0$ and $f_0(x) = d$.

Now, $\|z\| = \|\lambda x + y\| = |\lambda| \cdot \|x + y/\lambda\| \geq |\lambda| \cdot d = |f_0(z)|$, since $(-y/\lambda) \in L$ and the distance from x to $(-y/\lambda)$ is at least d. This shows that $\|f_0\| \leq 1$. Moreover, there exists $y_n \in L$ so that $\|x+y_n\| \to d$. Hence $d = |f_0(x+y_n)| \leq \|f_0\| \cdot \|x+y_n\| \to d.\|f_0\|$, implies $\|f_0\| \geq 1$. Thus $\|f_0\| = 1$.

By Hahn-Banach theorem 4.1.1, there exists an extension f of f_0 to all of X, with $\|f_0\| = \|f\|$. Therefore, we have $f \in X^*$ such that $\|f\| = 1$, $f|_{L_1} = f_0$, which means $f(L) = 0$ and $f(x) = d$. □

Corresponding to any subspace L of a normed space X, we associate a subspace of X^*, namely, the ***annihilator*** $L^\perp$ of L, which is defined by

$$L^\perp = \{f \in X^* \ : \ f(L) = 0\}.$$

It is clear that $L^\perp$ is a closed subspace of $X^\perp$.

Now, if we start with a subspace F of X^*, then the annihilator $F^\perp$ is a closed subspace of $(X^*)^* := X^{**}$. Corresponding

to this F, we consider the subspace $F_\perp$ of X, defined by

$$F_\perp = \{x \in X \ : \ f(x) = 0, \text{for all } f \in F\} = \bigcap_{f \in F} \ker f.$$

For any closed subspace L of a normed space X, we have the following result.

Corollary 4.1.7. *Let L be a closed subspace of X. Consider the subspaces $L^\perp \hookrightarrow X^*$ and $(L^\perp)_\perp = \{x \in X \ : \ f(x) = 0, \text{for all } f \in L^\perp\}$. Then $(L^\perp)_\perp = L$.*

Proof. If $x \in L$, then $f(x) = 0$ for all $f \in L^\perp$, so that $x \in \bigcap_{f \in L^\perp} \ker f$. Thus, $L \subset (L^\perp)_\perp$.

Now, if $x \notin L$, we have $\text{dist}(x, L) = d > 0$, as L is closed. By Corollary 4.1.6, there exists f such that $f(L) = 0$, i.e., $f \in L^\perp$ and $f(x) \neq 0$. Therefore, $x \notin (L^\perp)_\perp$. This implies $(L^\perp)_\perp \subset L$. Thus, $(L^\perp)_\perp = L$. □

Remark 4.1.8. Note that for any subspace L, not necessarily closed, $(L^\perp)_\perp = \bar{L}$.

Corollary 4.1.9. *Let X be a normed space and let $\{x_1, x_2, ..., x_n\}$ be a linearly independent subset of X. Then, there exists*

$f_1, f_2, ..., f_n \in X^*$ such that $f_i(x_j) = \delta_{ij}$, where δ_{ij} is the Kronecker's delta function,

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

and every $x \in span\{x_i\}_1^n$ can be expressed as $x = \sum_{i=1}^{n} f_i(x)x_i$.

Proof. For any fixed i_0, let $L_{i_0} = \text{span}\{x_i\}_{i \neq i_0}$. As every finite dimensional subspace of a normed space is closed (Try to prove this!), L_{i_0} is a closed subspace of X and $x_{i_0} \notin L_{i_0}$.

By Corollary 4.1.6, there exists $f_{i_0} \in X^*$ such that $f_{i_0}(L_{i_0}) = 0, f_{i_0}(x_{i_0}) = 1$. Thus, if $x = \sum a_i x_i$,then $a_i = f_i(x)$. This completes the proof. □

4.2 Reflexive Spaces

Let X be a normed space and let X^* be its dual space. We have noted that X^* is always a Banach space. We now consider the dual space $(X^*)^*$ of X^*, which we denote by X^{**}. We may identify X with a subspace of X^{**} in the following way: For

$x \in X$, consider the functional $i(x) \in X^{**}$ such that for every $f \in X^*$, $i(x)(f) = f(x.)$ Then clearly, $i : X \to X^{**}$ defines a one-to-one linear map. Also,

$$|i(x)(f)| \leq \|f\| \cdot \|x\|.$$

This implies that,

$$\|i(x)\|^{**} \quad = \sup_{f \neq 0} \quad \frac{|i(x)(f)|}{\|f\|} \quad = \sup_{f \neq 0} \frac{|f(x)|}{\|f\|} \leq \|x\|.$$

But, Corollaries 4.1.2, and 4.1.3, shows that the last supremum is attained and for any $x \in X$, there exists $f \neq 0$ such that $|f(x)| = \|x\| \cdot \|f\|$. Therefore, $\|i(x)\|^{**} = \|x\|$. Thus, the map $i : X \to X^{**}$ is an isometry into.

We say that a normed space X is a ***reflexive space*** if the map $i : X \to X^{**}$ is onto. Thus, in case of reflexive spaces, the spaces X and X^{**} are isometric, and we can identify them in a natural way. For instance, in case of reflexive spaces, we do not need to distinguish between $F^{\perp}$ and $F_{\perp}$, and in this case, the conclusion of Corollary 4.1.7 becomes $(L^{\perp})^{\perp} = L$.

We now consider the dual spaces of some normed spaces.

Example 4.2.1. Consider the space c_0 of null sequences (i.e., the set of all scalar sequences that converges to 0) equipped with the norm $\|x\| = \sup_i |a_i|$, for $x = (a_i)$. Fix a sequence $b = (b_i) \in \ell_1$ and using it, define a linear functional on c_0 as $f(x) = \sum_i a_i b_i$ for $x = (a_i) \in c_0$. Then

$$|f(x)| \leq \sum_i |a_i b_i| \leq \sup_i |a_i| \cdot \sum_i |b_i| = \|x\|_{c_0} \cdot \|b\|_{\ell_1}.$$

Therefore, $\|f\|_{c_0^*} \leq \|b\|_{\ell_1}$.

Thus, starting with $b = (b_i) \in \ell_1$, we get $f \in c_0^*$, and for this f, we have $f(e_n) = b_n$, for $n = 1, 2, ...$, where $e_n = (0, 0, ..., 0, 1, 0, ...) \in c_0$, 1 occurs only at the n^{th} place.

Similarly, starting with a $f \in c_0^*$, setting $b_n = f(e_n)$ for $n = 1, 2, ...$, we get $b = (b_n)_{n \in \mathbb{N}} \in \ell_1$. To see this, for each $n = 1, 2, ...$, consider the sequence

$$\begin{aligned} y_n &= (e^{-i \arg b_1}, e^{-i \arg b_2}, ..., e^{-i \arg b_n}, 0, 0, ...) \\ &= \sum_{k=1}^{n} e^{-i \arg b_k} e_k \end{aligned}$$

Then $y_n \in c_0$ and $\|y_n\|_{c_0} = 1$ for all $n = 1, 2, ...$ By the definition of norm of a functional,

$$\begin{aligned} \|f\|_{c_0^*} &\geq |f(y_n)| \\ &\geq f(y_n) \\ &= f(\sum_{k=1}^{n} e^{-i \arg b_k} e_k) \\ &= \sum_{k=1}^{n} e^{-i \ arg \ b_k} b_k \\ &= \sum_{k=1}^{n} |b_k|. \end{aligned}$$

(Here, we used the idea, if $z = re^{i\theta}$, then $e^{-i\theta}z = |z|$.) This is true for any $n = 1, 2, 3, ...$ This implies $\|f\|_{c_0^*} \geq \|b\|_{\ell_1}$. This shows that $b \in \ell_1$.

Thus, any $f \in c_0^*$ is identified with an element $b \in \ell_1$ and $\|f\|_{c_0^*} \geq \|b\|_{\ell_1}$.

For any such $b \in \ell_1$, we proved above that $\|f\|_{c_0^*} \leq \|b\|_{\ell_1}$. Combining these inequalities we obtain, $\|f\|_{c_0^*} = \|b\|_{\ell_1}$. Hence we can identify the dual space of c_0 as ℓ_1, and this identification is an isometry.(i.e., preserving the norms.)

In a similar manner, one can prove that $\ell_1^* = \ell_\infty$. Thus c_0 is not a reflexive space.

Example 4.2.2. Consider the space of $p-$summable scalar sequences ℓ_p, with $1 < p < \infty$. We claim that $(\ell_p)^* = \ell_q$, where $\frac{1}{p} + \frac{1}{q} = 1$. First let $b = (b_i) \in \ell_q$. Then b defines a linear functional on ℓ_p by the formula

$$f(x) = \sum_{i=1}^{\infty} a_i b_i,$$

for any $x = (a_i) \in \ell_p$. Also, by Hölder's inequality,

$$|f(x)| \leq \sum |a_i b_i| \leq (\sum |a_i|^p)^{1/p} \cdot (\sum |b_i|^q)^{1/q}.$$

$$\text{i.e., } |f(x)| \leq \|b\|_{\ell_q} \cdot \|x\|_{\ell_p}.$$

This implies, $\|f\|_{\ell_p^*} \leq \|b\|_{\ell_q}$, and hence $f \in \ell_p^*$.

Thus, corresponding to $b = (b_i) \in \ell_q$, we identified a bounded linear functional $f \in \ell_p^*$, such that $f(e_n) = b_n$ for $n = 1, 2, ...$, where $e_n = (0, 0, ..., 0, 1, 0, ...) \in \ell_p$, 1 occurs only at the n^{th} place.

Now, start with $f \in \ell_p^*$, and set $b_n = f(e_n)$.

Then $b = (b_n) \in \ell_q$. To see this, set $y_n = \sum_{k=1}^{n} e^{-i \arg b_k} |b_k|^{q-1} e_k$. Then $y_n \in \ell_p$ and

$$\|y_n\|_{\ell_p} = (\sum_{1}^{n} |b_k|^{(q-1)p})^{1/p} = (\sum_{1}^{n} |b_k|^q)^{1/p},$$

since $(q-1)p = q$. (To see this recall that $1/p + 1/q = 1$, so that $q = \frac{p}{p-1}$.)

Also,

$$\begin{aligned} \|f\|_{\ell_p^*} \|y_n\|_{\ell_p} &\geq |f(y_n)| \\ &= (\sum_{k=1}^{n} |b_k|^q) \\ &= (\sum_{k=1}^{n} |b_k|^q)^{1/p} \cdot (\sum_{k=1}^{n} |b_k|^q)^{1/q} \\ &= \|y_n\|_{\ell_p} \cdot (\sum_{k=1}^{n} |b_k|^q)^{1/q} \end{aligned}$$

Letting $n \to \infty$, we get $\|f\|_{\ell_p^*} \|y_n\|_{\ell_p} \geq \|y_n\|_{\ell_p} \cdot \|b\|_{\ell_q}$, which implies, $\|f\|_{\ell_p^*} \geq \|b\|_{\ell_q}$, showing that $b = (b_i) \in \ell_q$.

Thus, any $f \in \ell_p^*$ is identified with an element $b \in \ell_q$ and

$\|f\|_{\ell_p^*} \geq \|b\|_{\ell_q}$.

For any such $b \in \ell_q$, we proved above that $\|f\|_{\ell_p^*} \leq \|b\|_{\ell_q}$. Combining these inequalities we get $\|f\|_{\ell_p^*} = \|b\|_{\ell_q}$. Therefore, we can isometrically identify $(\ell_p)^*$ with the space ℓ_q. This shows that all the ℓ_p-spaces, $1 < p < \infty$ are reflexive spaces.

A similar result holds also in case of function spaces L^p, namely, $L_p^* = L_q$, for $1 < p < \infty$.

Chapter 5

Bounded Linear Operators

In this chapter, we concentrate on bounded linear maps between normed spaces. Also we introduce a norm on the linear space of all bounded linear maps from a normed space to another.

We note that a linear map is a bounded operator if and only if it is continuous.

5.1 The Space of Bounded Linear Operators

Let X and Y be Banach spaces and let $T : X \to Y$ be a linear map (usually called an operator) defined on X. Then, T is said to be ***bounded*** if there exists $C > 0$ such that $\|Tx\|_Y \leq C\|x\|_X$ for all $x \in X$. Thus, *a linear map $T : X \to Y$ is bounded if T maps bounded sets of X into bounded sets of Y.*

If T is bounded, define

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}$$

Clearly, if $\|Tx\| \leq C$ for every $x \in X$ with $\|x\| = 1$, then $\|T\| \leq C$. (Since, $\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{x \neq 0} \|T\big(\frac{x}{\|x\|}\big)\|$, here $\frac{x}{\|x\|}$ is of norm 1, therefore we have $\|T(\frac{x}{\|x\|})\| \leq C$)

Moreover, the family of linear bounded operators is a linear space with respect to the addition and the multiplication of operators by scalars (Indeed, if T is such that $\|Tx\| \leq C_1\|x\|$ and U is such that $\|Ux\| \leq C_2\|x\|$, then $\|(\alpha T + U)x\| \leq |\alpha|\|Tx\| + \|Ux\| \leq (|\alpha|C_1 + C_2)\|x\|$). We now check that $\|T\|$ defines a

norm.

1. Clearly, $\|T\| \geq 0$
2. $T = 0 \Rightarrow \|T\| = 0$, if $\|T\| = 0$ then $T = 0$. Suppose if possible $T \neq 0$, then there exists an $x \neq 0$ such that $Tx = c > 0$, then $T(\frac{x}{\|x\|}) = \frac{c}{\|x\|} > 0$ contradicts that $\|T\| = 0$
3. $$\begin{aligned} \|T+U\| &= \sup_{x\neq 0} \frac{\|(T+U)x\|}{\|x\|} \\ &\leq \sup_{x\neq 0} \frac{\|Tx\| + \|Ux\|}{\|x\|} \\ &\leq \|T\| + \|U\| \end{aligned}$$
4. $\|\alpha T\| = \sup_{x\neq 0} \frac{\|(\alpha T)x\|}{\|x\|} = \sup_{x\neq 0} \frac{|\alpha|\|Tx\|}{\|x\|} = |\alpha|\|T\|$

Thus, $\|T\| = \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|}$ defines a norm, and is called the ***operator norm***.

We write $L(X \to Y)$ or $L(X, Y)$ to denote the linear space of bounded operators with the above norm. From the definition

of the norm $\|T\|$, it follows that for every $x \in X$ we have

$$\|Tx\| \leq \|T\| \cdot \|x\|.$$

If H is a Hilbert space, then for every bounded linear operator $A : H \to H$ we have

$$\|A\| = \sup\{|\langle Ax, y\rangle| : \|x\| \leq 1, \|y\| \leq 1\}$$

To see this, let $t = \sup\{|\langle Ax, y\rangle| : \|x\| \leq 1, \|y\| \leq 1\}$. Now by applying the Cauchy-Schwartz inequality, we get
$|\langle Ax, y\rangle| \leq \|Ax\|\|y\| \leq \|A\|\|x\|\|y\| \Rightarrow t \leq \|A\|$.

Now we need to show $t \geq \|A\| = \sup_{x \neq 0} \dfrac{\|Ax\|}{\|x\|}$.

It is sufficient to show that $\|Ax\| \leq t\|x\|$ whenever $x \neq 0$. Now, use the fact that
$\|Ax\|^2 = \|x\|\|Ax\|\langle A(\dfrac{x}{\|x\|}), \dfrac{Ax}{\|Ax\|}\rangle \leq t\|x\|\|Ax\|$, we get
$\|Ax\| \leq t\|x\|$.

Theorem 5.1.1. *Let X be a normed space and let Y be a complete normed space. Then $L(X, Y)$ is a Banach space (i.e., it is a complete normed space).*

Proof. Let $\{A_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $L(X,Y)$, so that for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $m, n \geq N$ we have

$$\|A_n - A_m\| \leq \epsilon$$

This implies that for every $x \in X$ and $m, n \geq N$

$$\begin{aligned} \|A_n(x) - A_m(x)\|_Y &= \|(A_n - A_m)(x)\|_Y \\ &\leq \|A_n - A_m\| \cdot \|x\| \\ &\leq \epsilon \|x\|. \end{aligned}$$

Therefore, for all $x \in X$, the sequence $\{A_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence in Y.

Since Y is a Banach space, this sequence has a limit, say $A(x)$ in Y, and thus, we define for all $x \in X$, $A(x) = \lim_{n\to\infty} A_n(x)$. Then A is a linear operator and it is also bounded, since

$$\|A(x)\|_Y \leq \sup_{n\in\mathbb{N}} \|A_n(x)\| \leq \|x\|_X \cdot \sup_{n\in\mathbb{N}} \|A_n\|,$$

hence

$$\|A\| \leq \sup_{n\in\mathbb{N}} \|A_n\|.$$

Here, we use the fact that a Cauchy sequence is bounded. Here

both A_n and $A_n x$ are Cauchy and therefore both supremums are finite. Also $\|Ax\| = \|\lim A_n x\| = \lim \|A_n(x)\|$.
Thus, $A \in L(X, Y)$.

Now it remains to show that $A_n \to A$.
Since $\{A_n\}$ is a Cauchy sequence, for any $\epsilon > 0$ there exists N such that for any $m, n > N$ we have $\|A_n - A_m\| < \epsilon$.

This means that for such m, n and for any x such that $\|x\| \leq 1$ we have $\|A_m x - A_n x\| < \epsilon$. Letting m tends to infinity, we obtain $\|Ax - A_n x\| \leq \epsilon$ for any x with $\|x\| \leq 1$.

Hence $\|A - A_n\| \leq \epsilon$ for any $n > N$, which implies

$$\lim_{n\to\infty} \|A - A_n\| = 0.$$

□

Next, we note that ***a linear map*** A ***is a bounded operator if and only if*** A ***is a continuous operator(i.e.,***$Ax_n - Ax \to 0$ ***for any sequence*** $x_n \to x$***).***

First, assume that A is a bounded operator. We have to show that $x_n \to x \Rightarrow Ax_n \to Ax$.
$\|Ax_n - Ax\| = \|A(x_n - x)\| \leq \|A\|\|x_n - x\| \to 0$ since $x_n \to x$.

Therefore, A is continuous.

Conversely suppose that A is continuous. Then A is continuous at 0 also, which implies $\exists \delta > 0$ such that $\|x\| < \delta \Rightarrow \|Ax\| < 1$.
Therefore, whenever $y \neq 0, \|A(\delta y/\|y\|)\| < 1 \Rightarrow \|Ay\| \leq \|y\|/\delta$, when $y = 0$ trivial. Therefore A is a bounded operator. Thus, a linear map A is a bounded operator if and only if A is a continuous operator.

Also, note that, ***for any bounded linear operator*** A, ***the set ker*** $A = \{x : Ax = 0\}$ ***is a closed subspace.***

Indeed, if x_n be a sequence in ker A, then $Ax_n = 0$ for all x_n. If $x_n \to x$, then $0 = Ax_n \to Ax = 0$. Therefore, $x \in \ker A$.

In Chapter 4, we have observed that for any normed space X, the dual space X^* is complete. Indeed, take Y to be the field $\mathbb{R}$ or $\mathbb{C}$ (depending over which field, the space X is considered). Then $L(X, Y) = X^*$.

Next, let X, Y and Z be Banach spaces and let $A \in L(X, Y), B \in L(Y, Z)$. Then $BA \in L(X, Z)$ and

$$\|BA\| \leq \|B\|\|A\|.$$

To see this, suppose $x_n \to x \Rightarrow Ax_n \to Ax \Rightarrow B(Ax_n) \to B(Ax)$. Therefore, $BA \in L(X, Z)$.
Also, $\|B(Ax)\| \leq \|B\|\|Ax\| \leq \|B\|\|A\|\|x\|$.
Therefore, $\|BA\| \leq \|B\|\|A\|$.

Now we illustrate some examples of linear operators.

Example 5.1.2. Consider the space $C[0, 1]$. For $f \in C[0, 1]$, define

$$Af = \int_0^1 K(t, \tau) f(\tau) d\tau,$$

where K is a continuous function of two variables. The operator A is linear and

$$\|Af\|_{C[0,1]} \leq \max|f|.\max_t \int_0^1 |K(t, \tau)| d\tau.$$

So, $\|A\| \leq \max_t \int_0^1 |K(t, \tau)| d\tau$.

It can be shown that

$$\|A\| = \max_t \int_0^1 |K(t, \tau)| d\tau.$$

Example 5.1.3. In $L_2[0, 1]$ and for an element $K(t, \tau)$ of $L_2(I^2)$,

where $I^2 = [0,1] \times [0,1]$, define the operator

$$K : L_2[0,1] \mapsto L_2[0,1]$$

by

$$(Kf)(t) = \int_0^1 K(t,\tau) f(\tau) d\tau.$$

The function of two variables $K(t,\tau)$ is called the *kernal*(or the *kernal function*) of the operator K. One may easily check using the Cauchy-Schwartz inequality that the norm of the operator K(denoted here by $\|K\|_{op}$ to avoid confusion) satisfies

$$\|K\|_{op} \leq \|K(t,\tau)\|_{L_2(I^2)}$$

Indeed, we have

$$\int_0^1 |K(t,\tau) f(\tau)| d\tau \leq \left(\int_0^1 |K(t,\tau)|^2 d\tau \right)^{1/2} \left(\int_0^1 |f(\tau)|^2 d\tau \right)^{1/2}.$$

Hence,

$$\begin{aligned}\|Kf\|^2 &\leq \int_0^1 \left(\int_0^1 |K(t,\tau)f(\tau)|d\tau \right)^2 dt \\ &\leq \|f\|^2 \int_0^1 \int_0^1 |K(t,\tau)|^2 d\tau dt\end{aligned}$$

and it follows that,

$$\|K\|_{op} \leq \|K(t,\tau)\|_{L_2(I^2)}.$$

Example 5.1.4. Let $k(t)$ be a continuous function on $[a,b]$. In $L_2[a,b]$ define the operator A by $Af = k(t) \cdot f(t)$. Then A is a bounded linear operator and

$$\|A\| := M := \max_{a \leq t \leq b} |k(t)|.$$

It is immediate that $\|A\| \leq M$. To show that, $\|A\| = M$, suppose $M = |k(t_0)|$. Define a sequence $\{x_n\}$ by

$$x_n(t) = \begin{cases} \sqrt{n/2}, & t \in [t_0 - \frac{1}{n}, t_0 + \frac{1}{n}], \\ 0, & \text{otherwise.} \end{cases}$$

Now

$$\|x_n\|^2 = \int_{t_0-1/n}^{t_0+1/n} \frac{n}{2} = 1,$$

and from the continuity of $k(t)$ at t_0, it follows that

$$\|A\|^2 \geq \|Ax_n\|^2 = \frac{n}{2} \int_{t_0-1/n}^{t_0+1/n} |k(t)|^2 dt \to |k(t_0)|^2.$$

Hence

$$\|A\| \geq \|k(t_0)| = M.$$

Example 5.1.5. The shift operator in ℓ_2 defined by $Tx = (0, a_1, \dots, a_n, \dots)$ for $(a_n) \in \ell_2$ satisfies $\|Tx\| = \|x\|$ for every x and therefore $\|T\| = 1$

Example 5.1.6. Let $(\alpha_{ij})_{i,j=1}^{\infty}$ be an infinite matrix and let $K^2 = \sum_{i,j=1}^{\infty} |\alpha_{ij}|^2 < \infty$. Then the operator A defined in ℓ_2 by

$$A\left((\alpha)_{i=1}^{\infty}\right) = (\beta_i)$$

where

$$\beta_i = \sum_{j=1}^{\infty} \alpha_{ij}\alpha_j$$

for $i \in \mathbb{N}$ is a bounded linear operator .

To see this one has to check that $\|A(\alpha_i)\|_{\ell_2} \leq K.\|(\alpha_i)\|_{\ell_2}$(again, one should use the Cauchy-Schwartz inequality).

Example 5.1.7. Let H be a separable Hilbert space and let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis. Let $A : H \to H$ be a bounded linear operator.

We have $Ax = \sum_{i\geq 1} \langle Ax, e_i\rangle e_i$ and $x = \sum_{j\geq 1} x_j e_j$, where $x_j = \langle x, e_j\rangle$.
From the second relation, since A is a bounded operator, we get $Ax = \sum_{j\geq 1}\langle x, e_j\rangle Ae_j$. Substituting in the first relation, we obtain

$$Ax = \sum_{i\geq 1}\Big\langle \sum_{j\geq 1}\langle x, e_j\rangle Ae_j, e_i\Big\rangle e_i = \sum_{i\geq 1}\sum_{j\geq 1}\langle Ae_j, e_i\rangle\langle x, e_j\rangle e_i.$$

Set $a_{ij} = \langle Ae_j, e_i\rangle$. Thus if $x = \sum_{i\geq 1} x_i e_i$, $y = \sum_{i\geq 1} y_i e_i$ and $y = Ax$, we get that $y_i = \sum_{j\geq 1} a_{ij}x_j$.
Hence, the sequence (α_i) is mapped by A to $\Big(\beta_i = \sum_j^r \langle Ae_j, e_i\rangle\alpha_i\Big)_{i=1}^{\infty}$.
Consequently, the operator A is defined by the matrix $\big(\langle Ae_j, e_i\rangle\big)_{i,j}$.
However, the example 5.1.6 may not be applicable, since the condition $\sum |\alpha_{ij}|^2 < \infty$ usually is not satisfied, for instance, consider the example of the identity operator $I : H \to H$.

5.2 Compact Operators

In this section, we introduce the compact linear maps, which are very important in applications. For instance, they play a central role in the theory of integral equations and in various problems of mathematical physics. Their theory served as a model for the early work in functional analysis. Their properties closely resembles those of operators on finite dimensional spaces. Compact linear maps are defined as follows:

Definition 5.2.1. A linear operator $A : X \to Y$ is called a ***compact operator*** if and only if for every bounded sequence $x_n \in X$ the sequence $\{Ax_n\}$ has a Cauchy subsequence.

Clearly, a compact operator must be bounded (and hence is continuous) because the image of the unit ball of X must be a bounded set in Y(otherwise we can easily build a sequence x_n inside the ball of X and such that $\|Ax_n\| \to \infty$, thus unbounded and hence without a Cauchy subsequence).

In order to work with compact operators, we should first understand well, the notion of a compact set.

A set $K \subseteq X$ is called a ***compact set*** if and only if for every

sequence $x_n \in K$ there exists an $x \in K$ and a subsequence x_{n_i} of x_n so that $x_{n_i} \to x$.

K is said to be ***relatively compact*** (or ***precompact***) if every sequence $x_n \in K$ has a Cauchy subsequence x_{n_i}. For example, if X is complete and K is relatively compact, then the closure $\overline{K}$ is compact.
From the **Heine- Borel theorem** of Real Analysis, it follows that ***any bounded set in*** $\mathbb{R}^n$ ***is relatively compact.***

We now state an important result due to **Arzela** .

Theorem 5.2.2. ***(Arzela)*** *Let $M \subseteq C[a,b]$. M is relatively compact (in $C[a,b]$) if and only if M is*
(i) uniformly bounded(i.e., it is a bounded set in $C[a,b]$) and
(ii) equicontinuous: for any $\epsilon > 0$ there exists $\delta > 0$ such that $|t_1 - t_2| < \delta$ implies $|f(t_1) - f(t_2)| < \epsilon$ for any $f \in M$.

Notice that if F is any metric precompact space, then the same theorem is true for $M \subseteq C(F)$.

The proofs of the facts given in the following remark are left as exercises.

Remark 5.2.3. (i) A subset of a precompact set is precompact.
(ii) For a bounded linear operator A if the image of the unit ball

is precompact, then the image of any ball is precompact.
(iii)An operator $A : X \to Y$ is compact if and only if the image of the unit ball of X is a precompact set in Y.

Example 5.2.4. Let C_1 and C_2 be positive constants and let

$$M = \{x(t) \in C[a,b] : \ |x(t)| < C_1 \text{ and } |x'(t)| < C_2\}.$$

Then M is relatively compact. (To see this, use Arzela's theorem: $|x'(t)| < C_2$ implies the uniform continuity: $(x(t_1) - x(t_2))/(t_1 - t_2) = x'(\theta)$ for $t_1 < \theta < t_2$, and therefore, $|x(t_1) - x(t_2)| \leq C_2 \cdot |t_1 - t_2|$)

Example 5.2.5. The operator $Ax = \int_0^t x(\tau)d\tau$ on $C[0,1]$ is a compact operator (use the previous example).

Example 5.2.6. Let $C_1[a,b]$ be a space of continuous functions on $[a,b]$ with continuous derivative, and let

$$\|x(t)\|_{C_1} = \max_t |x(t)| + \max_t |x'(t)|.$$

Consider the *embedding operator* $A : C_1[a,b] \mapsto C[a,b]$ with $Ax = x$. Then, A is a compact operator. The unit ball of $C_1[a,b]$ is a precompact set in $C[a,b]$ by Arzela's theorem.

Example 5.2.7. Let $K(t,\tau)$ be a continuous function of two variables on $[0,1]\times[0,1]$.
Let us prove that the operator $Kx = \int_0^1 K(t,\tau)x(\tau)d\tau : C[0,1] \to C[0,1]$ is a compact operator. Let $M = \{x(t)\}$ be a bounded set of functions from $C[0,1]$ such that $\|x\| \leq r$. It is clear that the functions $y(t) = \int_0^1 K(t,\tau)x(\tau)d\tau$, where $x(t)$ is a function from M, are uniformly bounded. Indeed, if $R = \max\limits_{t,\tau} K(t,\tau)$, then $|y(t)| \leq Rr$. It means that the operator K is bounded, which was already proved before.

Next, the functions $y(t)$ are uniformly continuous. Indeed, given $\epsilon > 0$ by the uniform continuity of the kernel $K(t,\tau)$ there exists $\delta > 0$ such that $|K(t_1,\tau) - K(t_2,\tau)| < \epsilon/r$ for $|t_1 - t_2| < \delta$ and any $\tau \in [0,1]$. It follows that $|t_1 - t_2| < \delta$ implies

$$|y(t_1) - y(t_2)| \leq \int_0^1 |K(t_1,\tau) - K(t_2,\tau)||x(\tau)|d\tau < \epsilon$$

for any function $y(t)$ from the set under consideration, which means that the set is equicontinuous.

Thus, by the Arzela theorem we conclude that the set $\{y(t)\}$ is relatively compact in $C[0,1]$. Hence the operator K is compact.

To build more examples of compact operators, we need some properties of compact sets and operators.

Definition 5.2.8. Let X be a metric space and let $A \subseteq X$. A set $\mathcal{A} \subset X$ is called an $\epsilon - net$ of A if and only if for all $x \in A$ there exists $y \in \mathcal{A}$ such that the distance from x to y is less than ϵ.

Lemma 5.2.9. *M is relatively compact if and only if for every $\epsilon > 0$ there exists a finite $\epsilon - net$ in M*

Proof. Assuming that there exists $\epsilon_0 > 0$ such that there is no finite $\epsilon_0 - net$ of M, take any $x_1 \in M$ and choose $x_2 \in M$ such that $\|x_1 - x_2\| \geq \epsilon_0$. If $\{x_i\}_1^{n-1}$ are already chosen, we can choose $x_n \in M$ such that $\|x_n - x_1\|, \|x_n - x_2\|, \ldots, \|x_n - x_{n-1}\| \geq \epsilon_0$. Such an x_n exists(for every n), otherwise $\{x_1, x_2, \ldots x_{n-1}\}$ is a finite $\epsilon_0 - net$ of M. Therefore no subsequence of $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence, which means that M is not precompact, a contradiction.

For the converse, for all $k \in N$ take an $\epsilon_k \leq 1/2k$. M has a finite $\epsilon_k - net$ for every k(by the assumption). Let $\{x_n\}_{n=1}^{\infty} \in M$ and consider a finite $\epsilon_1 - net$. It divides the sequence between a finite number of balls around the net points. So, there is atleast

one ball which contains an infinite subsequence from the original sequence. Let us denote the subsequence contained within this ball by $\{x_n^{(1)}\}_{n=1}^{\infty}$. In similar way, an $\epsilon_k - net$ divides the sequence $\{x_n^{(k-1)}\}_{n=1}^{\infty}$ between a finite number of balls and there is one ball which contains an infinite subsequence of the previously chosen sequence $\{x_n^{(k-1)}\}$. Let us call it $\{x_n^{(k)}\}_{n=1}^{\infty}$. We know that on this step the ball's radius is less than ϵ_k; hence for all $m, n \in N$ we have

$$\|x_m^{(k)} - x_n^{(k)}\| \leq 2\epsilon_k \leq \frac{1}{k}.$$

The subsequence $\{x_n^{(n)}\}_{n=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Indeed, for all $\eta \geq 0$ we can choose $\nu \in N$ such that $1/\nu \leq \eta$ and then for all $m, n \geq \nu$ we have

$$\|x_n^{(n)} - x_m^{(m)}\| \leq \frac{1}{\nu} \leq \eta$$

since both vectors belong to $\{x_i^{(\nu)}\}$. This implies that M is relatively compact. □

We now list some properties of the space of compact operators as a theorem.

Theorem 5.2.10. *The set $K(X \to Y)$ of compact operators*

from X to Y satisfy
(i) $K(X \to Y)$ is a linear subspace of $L(X \to Y)$.
(ii) If $A \in K(X \to Y), B \in L(Z \to X), C \in L(Y \to Z)$, then $AB \in K(Z \to Y), CA \in K(X \to Z)$. In the case $X = Y$ it means that $K(X) \equiv K(X \to X)$ is a two-sided ideal of $L(X \to X) \equiv L(X)$.
(iii) $K(X \to Y)$ is a closed subspace of $L(X \to Y)$.

Proof. (i) One can easily verify that $K(X \to Y)$ is a linear subspace of $L(X \to Y)$.

(ii) For the operator AB, let $\{z_n\}_{n=1}^{\infty}$ be a bounded sequence in Z. B is bounded; hence $\{B(z_n)\}_{n=1}^{\infty}$ is also bounded. A is compact; hence $\{A(B(z_n))\}_{n=1}^{\infty}$ has a converging subsequence and thus AB is compact.

For the operator BA, let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence in X. A is compact; hence the sequence $\{A(x_n)\}_{n=1}^{\infty}$ has a converging subsequence $\{A(x_{n_k})\}_{k=1}^{\infty}$. B is bounded, so it follows that the sequence $B(A(x_{n_k}))_{k=1}^{\infty}$ is also converging and thus BA is compact.

(iii) Let $A_n \to A$(i.e., $\|A_n - A\| \to 0$) and let $A_n \in K(X \to Y)$. In order to show that A is compact, it is enough to prove

that $A\mathcal{D}(X)$ (the image of the unit ball) is precompact. Thus, we have to find for all $\epsilon > 0$ a finite $\epsilon - net$ for $A\mathcal{D}(X)$.

For all $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $\|A_n - A\| < \epsilon/2$ and $A_n\mathcal{D}(X)$ is precompact (A_n is a compact operator).

Take an $\epsilon/2-net$ $\{y_i\}_1^N$ of $A_n\mathcal{D}(X)$. Then $\{y_i\}_1^N$ is an $\epsilon-net$ for $A\mathcal{D}(X)$.

Indeed, for any $y \in A\mathcal{D}(X)$ there is $x \in \mathcal{D}(X)$ such that $y = Ax$. Since $\{y_i\}_1^N$ is an $\epsilon/2 - net$ of $A_n\mathcal{D}(X)$, take y_{i_0} from the net such that $\|y_{i_0} - A_n x\| \leq \epsilon/2$.

Then $\|y - y_{i_0}\| \leq \|Ax - A_n x\| + \|A_n(x) - y_{i_0}\| \leq \epsilon$. □

5.3 Dual Operators

Let $A : X \to Y$ be a bounded operator and let $\phi \in Y^*$. Then $f(x) = \phi(Ax)$ for $x \in X$ defines a linear function on X.
Moreover,

$$|f(x)| \leq \|\phi\|_{Y^*} \cdot \|A\| \cdot \|x\|.$$

Thus, $f \in X^*$. Hence, we have a map $\phi \mapsto f := A^*\phi$. It is clear that this map $A^* : Y^* \mapsto X^*$ is linear. So, for every $x \in X$ and

$\phi \in Y^*$ we have

$$\phi(Ax) \equiv (A^*\phi)(x).$$

We check now that A^* is also a bounded operator:

$$\begin{aligned}
\|A\| &= \sup_{\|x\|=1} \|A(x)\|_Y \\
&= \sup_{\|x\|=1} \sup_{\|\phi\|=1} |\phi(A(x))| \\
&= \sup_{\|\phi\|=1} \sup_{\|x\|=1} |\phi(A(x))| \\
&= \sup_{\|\phi\|=1} \sup_{\|x\|=1} |(A^*\phi(x))| \\
&= \sup_{\|\phi\|=1} \|(A^*(\phi))\|_{X^*} \\
&= \|A^*\|.
\end{aligned}$$

We thus see that we always have $\|A\| = \|A\|^*$. The operator A^* is called the ***dual operator*** to the operator A. Note that $A^* : Y^* \to X^*$.

Let us introduce now more convenient notation to deal with duality and dual operators. Let X be a (normed) linear space and X^* be its dual. For $f \in X^*$ and $x \in X$ we write the action f on x in a symmetric way as $\langle x, f\rangle$ instead of $f(x)$(i.e., by

definition $\langle x, f\rangle = f(x)$.)

For a general linear space E and a space of functionals F on E (F may not coincide with E^*) we call $\langle x, f\rangle$ the "pairing" between E and F. We "copy" in this notation the inner product notation but now the first and the second arguments belong to different spaces. Applying this notation to the pairing between X and X^*, we can rewrite the definition of the dual operator A^* of an operator A in the following(symmetric) way:

For all $x \in X$ and for all $f \in Y^*$,

$$\langle Ax, f\rangle = \langle x, A^* f\rangle$$

Observe that if $A \in L(X, Y)$ and $B \in L(Y, Z)$, then $(BA)^* = A^*B^* : Z^* \mapsto X^*$
(To see this, note: $((A^*B^*)f)(x) = (A^*(B^*f))(x) = (B^*f)(Ax) = f(B(Ax)) = ((BA)^*f)(x))$

In the case of Hilbert spaces, for an operator $A \in L(H_1 \to H_2)$, the adjoint operator $A \in L(H_2 \to H_1)$ is defined via the relation

$$\langle Ax, y\rangle = \langle x, A^*y\rangle \quad \text{for all } x \in H_1, \text{ and for all } y \in H_2.$$

Example 5.3.1. Consider $K(t,\tau) \in L_2(I^2)(I=[0,1])$ and define the operator K on $L_2[0,1]$ by $(Kx)(t) = \int_0^1 K(t,\tau)x(\tau)d\tau$. Let us show that the adjoint operator K^* is given by $(K^*y)(t) = \int_0^1 \overline{K(\tau,t)}y(\tau)d\tau$.

Indeed, we have

$$\begin{aligned}\langle Kx, y\rangle &= \int_0^1 \left(\int_0^1 K(t,\tau)x(\tau)d\tau\right)\overline{y(t)}dt \\ &= \int_0^1 x(\tau)\overline{\left(\int_0^1 \overline{K(t,\tau)}y(t)dt\right)}d\tau \\ &= \langle x, y^*\rangle,\end{aligned}$$

where

$$y^*(\tau) = \int_0^1 \overline{K(t,\tau)}y(t)dt$$

Theorem 5.3.2. *A bounded operator $A : X \to Y$ is compact implies $A^* : Y^* \to X^*$ is compact.*

Proof. We show that $A^*\mathcal{D}(Y^*) = K \subset X^*$ is precompact. Let $f \in A^*\mathcal{D}(Y^*)$.

Then $f(x) = (A^*\phi)(x)$ for $x \in \mathcal{D}(X)$ and some $\phi \in \mathcal{D}(Y^*)$; that is, $f(x) = (A^*\phi)(x) = \phi(Ax)$.

We have

$$\|f\|_{X^*} = \sup_{\|x\|\leq 1} |f(x)| = \sup_{\|x\|\leq 1} |\phi(Ax)| = \sup_{y\in T} |\phi(y)|,$$

where $T = A\mathcal{D}(X)$ is precompact in Y since A is compact. Consider the collection of continuous functions on T, $M = \{\phi \in \mathcal{D}(Y^*)\}$.
For any $\phi \in \mathcal{D}(Y^*)$ we have $|\phi(y)| \leq \|A\|, y \in T$ and $|\phi(y_1) - \phi_2(y_2)| \leq \|y_1 - y_2\| < \epsilon$ for $\|y_1 - y_2\| < \epsilon$.
Thus by Arzela's theorem, M is precompact.

Let $\{f_n\}_{n=1}^{\infty}$ be a bounded sequence in $A^*\mathcal{D}(Y^*)$. From the above expression for $\|f\|_{X^*}$, it follows that the corresponding sequence of functions $\{\phi_n\}_{n=1}^{\infty}$ is also bounded.

Hence there exists a subsequence $\phi_{n'}$ such that
$\|\phi_{n'} - \phi_{n''}\|_{C(T)} \to 0$ as $n', n'' \to \infty$ which implies that
$\|f_{n'} - f_{n''}\|_{X^*} \to 0$ as $n', n'' \to \infty$.

So, $A \in K(X \to Y)$, where A is a compact operator, implies that $A^* \in K(Y^* \to X^*)$(i.e, A^* is also a compact operator). □

Remark 5.3.3. In the case where X and Y are Hilbert spaces, the above theorem may be proved without use of Arzela's theorem.

5.4 Finite Rank Operators

We call the number dim(Im A) the ***rank*** of the operator A and we write rank A=dim(ImA).

We say that the operator A has ***finite rank*** if rank $A < \infty$. For an example of an operator of finite rank, consider a finite number of elements $y_i \in Y$ and $f_i \in X^*$ where X and Y are Banach spaces and $i = 1, 2, \ldots, n$ for some $n \in \mathbb{N}$. Define the operator A by setting $Ax = \sum_1^n f_i(x)y_i$. This operator has rank no greater than n and it is a bounded operator since $\|A\| \leq \sum_i^n \|f_i\|.\|y_i\|$.

Indeed, we have $\|Ax\| = \|\sum_1^n f_i(x)y_i\| \leq \sum_1^n \|f_i(x)y_i\| \leq \sum_1^n \|f_i(x)\|\|y_i\| \leq \sum_1^n \|f_i\|\|x\|\|y_i\| = \|x\| \sum_1^n \|f_i\|\|y_i\|$.

Let us compute the dual operator A^*:

$$\phi(Ax) = \sum_1^n f_i(x)\phi(y_i) = \left(\sum_1^n \phi(y_i)f_i\right)(x)$$

So, $A^*\phi = \sum_1^n \phi(y_i)f_i$. Thus the operator A may be written as $A(.) = \sum_1^n f_i(.)y_i$ (where we put the "." in the place where the variable "x" should be applied) and its dual is $A^*(.) = \sum_1^n (.)(y_i)f_i$ (now "." is in the position of the variable $\phi \in Y^*$).

Every bounded operator of finite rank is a compact operator because it maps the unit ball to a bounded set in a finite dimensional subspace which is always precompact.

Example 5.4.1.

Compactness of the integral operator in L_2

Consider $K(t,\tau) \in L_2(I^2)$, where $I = [0,1]$, and let

$$C = \|K(t,\tau)\|_{L_2(I_2)} = \sqrt{\int_0^1 \int_0^1 \|K(t,\tau)\|^2 dt d\tau}.$$

Define the operator K on $L_2[0,1]$ by

$$Kx = \int_0^1 K(t,\tau)x(\tau)d\tau.$$

Then K is a compact operator in $L_2[0,1]$ and $\|K\|_{op} \leq C = \|K\|_{L_2(I^2)}$.

In order to prove this, we first recall that $\|K\|_{op} \leq \|K\|_{L_2(I^2)}$ (See Example 5.1.3.)

Let $\{\phi_i(t)\}_1^\infty$ be an orthonormal basis in $L_2([0,1])$. From Exercise 3.6, we see that $\{\phi_i(t)\phi_j(\tau)\}$ is an orthonormal basis

in $L^2[I^2]$; so

$$K(t,\tau) = \sum_{ij} a_{ij}\phi_i(t)\phi_j(\tau),$$

where the series converges in the sense of $L^2(I^2)$.

Let

$$K_n = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\phi_i(t)\phi_j(\tau).$$

Then $\|K_n - K\|_{L_2(I^2)} \to 0$ as $n \to \infty$; therefore, considering the operator

$$K_n x = \int_0^1 K_n(t,\tau)x(\tau)d\tau,$$

we also know that $\|K_n - K\|_{op} \to 0$ as $n \to \infty$. But, K_n is an operator of finite rank (rank $K_n \leq n$) and this means that K is approximable by finite rank operators, which are compact. This implies that the limit operator K is also compact.

5.5 Different Notions of Convergences in $L(X)$

In this section, we introduce different types of convergence in the space of operators $L(X) := L(X \to X)$, namely *norm convergence*, also called *uniform convergence*, *strong convergence* and *weak convergence.* The concept of weak convergence illustrates a basic principle of functional analysis, namely, the fact that the investigation of normed spaces is often related to that of their dual spaces.

Norm convergence, or ***Uniform convergence***, is defined by saying that the sequence of operators A_n ***converges in norm*** to the operator A (and we write $A_n \to A$) if $\|A_n - A\| \to 0$ as $n \to \infty$.

If X is a Banach space, then the space $L(X)$ is complete and so if $\{A_n\}$ is a Cauchy sequence with respect to the norm, then it always converges to a bounded operator.

Another important notion is that of ***strong convergence***: We say $A_n \to A$ ***strongly*** if for all $x \in X$ we have $A_n x \to Ax$.

For strong convergence we write $A_n \xrightarrow{s} A$.

We note here that if the sequence $\{A_n\}$ is Cauchy in the ***strong sense***, that is, for all $x \in X$ the sequence $A_n x$ is Cauchy in X, then there exists $A \in L(X)$ such that $A_n \to A$ strongly. This fact is an immediate consequence of the Banach-Steinhaus theorem, which is beyond the scope of our syllabus.

It is easy to see that ***strong convergence is weaker than norm convergence.*** That is, if $A_n \to A$ (converges in norm), then for any $x \in X$, $A_n x \to Ax$ since

$$\|A_n x - Ax\| \leq \|x\|.\|A_n - A\| \to 0.$$

However the opposite implication is not true. Let us give an example.
Consider the projections in $L_2[0,1]$ defined for $\epsilon_n > 0$ by

$$P_{\epsilon_n} f = \begin{cases} f(t), & \text{for } t < \epsilon_n \\ 0, & \text{for } t \geq \epsilon_n. \end{cases}$$

It is easy to see that $P_{\epsilon_n} f \to 0$ as $\epsilon_n \to 0$ for all $f \in L_2[0,1]$; that is, P_{ϵ_n} converges in the strong sense to zero. But $\|P_{\epsilon_n}\| = 1$, and consequently it does not converge in norm to zero.

(To see $\|P_{\epsilon_n}\| = 1$,, note that $\|P_{\epsilon_n} f\| \leq \|f\|$ by its definition. If we take $g = P_{\epsilon_n} f$, then $\|P_{\epsilon_n} g\| = \|P_{\epsilon_n} P_{\epsilon_n} f\| = \|P_{\epsilon_n} f\| = \|g\|$, for $f \in L_2[0,1]$, $P_{\epsilon_n} f \in L_2[0,1]$.)

Our third notion of convergence is that of ***weak convergence***. We say A_n ***converges weakly*** to A, and we write $A_n \xrightarrow{w} A$, if for all $x \in X$ and for all $f \in X^*$ we have $f(A_n x) \to f(Ax)$.

Again, in order to distinguish this notion from that of strong convergence, consider the example of the shift operator A in ℓ_2 (The shift operator in ℓ_2 defined by $Ax = (0, a_1, \dots, a_n, \dots)$ for $(a_n) \in \ell_2$, also we have $A^n(x) = (\overset{n \text{ times}}{0, \dots, 0}, a_1, a_2, \dots)$).

For this operator one can easily see that $f(A^n x) \to 0$ for every $f = \sum_1^\infty b_i e_i$ and $x = \sum_1^\infty a_i e_i$ since

$$f(A^n x) = \sum_{i=n+1}^{\infty} b_i a_{i-n}$$

and therefore

$$|f(A^n(x))| \leq \left(\sum_1^\infty |a_i|^2 \right)^{1/2} \left(\sum_{n+1}^\infty |b_i|^2 \right)^{1/2} \to 0,$$

as $n \to \infty$.

Thus, we have $A^n \xrightarrow{w} 0$, but $\|Ax\| = \|x\|$ for every x and so $\|A^n x\| = \|x\|$. Hence, A^n does not converge strongly to zero. (For any $x \neq 0, \|x\| \neq 0$. Hence $\|A^n x\|$ does not converge to zero$\Rightarrow$ $A^n x$ does not converge to zero.)

5.6 Invertible Operators

Let $A \in L(X)$. We call $B(:= A^{-1})$ the ***inverse operator*** of A if and only if $BA = Id$ and $AB = Id$, where Id is the identity operator.

In the finite dimensional case, the notion of the determinant implies that $BA = Id$ is enough to deduce that A is invertible and $A^{-1} = B$. The reason is that $detA \neq 0$ leads to a formula for A^{-1}.

In the infinite dimensional case, though, this is not the case. One may consider for example the case of the right shift operator in ℓ_2 : for $x = (x_i)_1^\infty$ let $Tx = (0, x_1, x_2, \dots)$. Define the left shift operator $\mathcal{U} : \ell_2 \mapsto \ell_2$ by $\mathcal{U}x = (x_2, x_3, \dots)$. Then $\mathcal{U}T = Id$ but $T\mathcal{U} \neq Id$ and ker$T\mathcal{U} \neq 0$.

We now list some properties of invertible operators.

1. If A and B are invertible operators, then AB is also invertible and its inverse satisfies $(AB)^{-1} = B^{-1}A^{-1}$.
   $((AB)(B^{-1}A^{-1}) = Id = (B^{-1}A^{-1})AB)$

2. If $\|A\| = q < 1$, then $(I - A)$ is invertible and $(I - A)^{-1} = \sum_0^\infty A^k$. Moreover, $\|(I - A)^{-1}\| \leq 1/(1 - \|A\|)$. Indeed, first note that $\|A\|^k \leq \|A\|^k \leq q^k \to 0$ as $k \to \infty$. The sequence $S_n = \sum_0^n A^k$ is Cauchy.

$$\begin{aligned}
\|S_n - S_m\| &= \|A^{m+1} + \dots A^n\| \\
&= \|A^{m+1}(I + A + \dots A^{n-m-1})\| \\
&\leq \|A^{m+1}\|\|(I + A + \dots A^{n-m-1})\| \\
&\leq \|A\|^{m+1}\|(I + A + \dots A^{n-m-1})\| \\
&\leq \frac{\|A\|^{m+1}}{1 - \|A\|} \to 0,
\end{aligned}$$

   since $\|A\| < 1 \Rightarrow \|A\|^{m+1} \to 0$, and therefore S_n converges.

   Moreover, $(I - A)S_n = (I - A)(I + A + \dots + A^n) = I - A^{n+1} \to I$ as $n \to \infty$ and $S_n(I - A) \to I$ as $n \to \infty$. Thus, $\lim S_n$ is the inverse operator of $(I - A)$

3. Let A be an invertible operator and let B be such that $\|A - B\| < 1/\|A^{-1}\|$. Then B is also invertible. Indeed, write $B = A(I - A^{-1}(A - B))$. The operator A is invertible and $I - A^{-1}(A - B)$ is invertible ($\|A^{-1}(A - B)\| \leq \|A^{-1}\|\|(A - B)\| < \|A^{-1}\|\frac{1}{\|A^{-1}\|} = 1$ and apply (2)). Hence, their product is invertible by (1).

We conclude this Chapter by stating one of the main theorems of functional analysis.

Theorem 5.6.1. ***(The Banach open map theorem)*** *Let X, Y be Banach spaces and let $A : X \to Y$ be a bounded linear operator which is one-to-one (i.e, ker A=0) and onto (i.e., Im A=Y). Then the formally defined inverse operator $A^{-1} : Y \to X$ is bounded.*

Recall that a map f from a metric space X to a metric space Y is said to be ***open*** if for every open set E in X, its image $f(E)$ is open in Y.

Note that a map $f : X \to Y$ is continuous if and only if for every open set E in Y, its inverse image $f^{-1}(E)$ is open in X.

Thus, a bijective map $f : X \to Y$ is open if and only if $f^{-1} : Y \to X$ is continuous.

Therefore, the Banach open map theorem states that if X and Y are Banach spaces, and if $A : X \to Y$ is a bijective, bounded linear map, then A is an open map.

References

Text Books (As per Syllabus)

1. Yuli Eidelman, Vitali Milman & Antonis Tsolomitis: *Functional Analysis: An Introduction*, Graduate Studies in Mathematics, Volume 66, American Mathematical Society, Providence, Rhode Island, 2004

Books For Further Reading

1. B.V. Limaye: *Functional Analysis*, New Age International Ltd., New Delhi, 1996.

2. E. Kreyzig: *Introductory Functional Analysis with Applications* , John Wiley and Sons, 1978.

3. M. Thamban Nair: *Functional Analysis: A First Course*, 2^{nd} Edition, PHI Learning Pvt. Ltd., 2021.

SYLLABUS FOR MSc MATHEMATICS (CBCSS) PG PROGRAMME

EFFECTIVE FROM 2019 ADMISSION ONWARDS

SEMESTER 3

MTH3C13: FUNCTIONAL ANALYSIS

No. of Credits: 4

No. of hours of Lectures/week: 5

TEXT : YULI EIDELMAN, VITALI MILMAN & ANTONIS TSOLOMITIS; FUNCTIONAL ANALYSIS AN INTRODUCTION; AMS, Providence, Rhode Island, 2004

Module 1

Linear Spaces; normed spaces;first examples: Linear spaces, Normed spaces;first exaples, Holder's inequality, Minkowski's ineaquality, Topological and geometric notions, Quotient normed space, Completeness;completion. [Chapter 1 Sections 1.1- 1.5]

Module 2

Hilbert spaces: Basic notions; first examples, Cauchy- Schwartz ineaquality and Hilbertian norm,Bessels ineaquality,Completesystems, Gram-Schmidt orthogonalization procedure, orthogonal bases, Parseval' identity; Projection; orthgonal decompositions; Separable case, The distance from a point to a convex set, Orthogonal decomposition; linear functionals; Linear functionals in a general linear space,Bounded linear functionals, Bounded linear functionals in a Hilbert space, An example of a non separable Hilbert space. [Chapter 2; Sections 2.1-2.3(omit Proposition 2.1. 15)]

Module 3

The dual space; The Hahn Banach Theorem and its first consequences, corollories of the Hahn Banach theorem, Examples of dual spaces. Bounded linear Operators; Completeness of the space of bounded linear operators, Examples of linear operators, Compact operators, Compact sets, The space of compact boperators, Dual operators ,Operators of finite rank, Compactness of the integral operators in L2, Convergence in the space of bounded operators, Invertible operators[Chapter3; Sections 3.1 , 3.2; Chapter4; Sections 4.1- 4.7]