# ADVANCED FUNCTIONAL ANALYSIS

## (MTH4C15)

## SELF LEARNING MATERIAL

### IV SEMESTER

### M.Sc. MATHEMATICS
### (2019 Admission)


## UNIVERSITY OF CALICUT
**SCHOOL OF DISTANCE EDUCATION**
**CALICUT UNIVERSITY P.O.**
**MALAPPURAM - 673 635, KERALA**

**190566**

# School of Distance Education
# University of Calicut

## Self Learning Material

## IV Semester

## M.Sc. Mathematics
**(2019 Admission)**

## MTH4C15 : Advanced Functional Analysis

*Prepared by:*

**Dr. K.M. SHINOJ**
**Assistant Professor**
**Department of Mathematics**
**St. Joseph's College, Devagiri, Calicut.**

***Scrutinized by:***

**ASWINI N.K.**
**Asst. Professor of Mathematics**
**SDE, University of Calicut.**

**DISCLAIMER**

"The author(s) shall be solely responsible for the content and views expressed in this book".

***Printed @ Calicut University Press***

## 5.1 CLASSIFICATION OF SPECTRUM

let $X$ be a Banach space and A: $X \to X$ be a bounded operator. A complex number $\lambda$ is called a regular point $A$ if $(A-\lambda I)^{-1}$: X $\to$ X exists and is a bounded operator. If $\lambda$ is not a regular point we call it a spectrum point.

The set of all spectrum points of A is called spectrum and A and is denoted by $\sigma(A)$.

**Result:** The set of all regular points is an open subset of $\mathbb{C}$.

**Proof:** Let $\lambda \in C$ be a regular point. So A - $\lambda$I is invertible and $(A-\lambda I)^{-1} \in L(X)$. We know Inv(X), the set of all invertible operators, is an open set in L(X). $\therefore$ there exists r>0 such that $U(A-\lambda I,r) \subset Inv(X)$.

Now let $\beta \in U(\lambda,r)$. ie $|\beta-\lambda)I|<r$.

$||(A-\lambda I) - (A - \beta I)|| = ||(\beta-\lambda|I|| = |\beta-\lambda)| < r$

$\therefore A - \beta I \in$ Inv X.

So $\beta$ is a regular point. $\therefore$ the set of regular points is an open set in C.

**Note:** $\sigma(A)$ is a closed set.

**Proposition:** Every $\lambda \in C$ with $|\lambda| > ||A||$ is a regular point of the operator A.

**Proof:** We have $A - \lambda I = \lambda(I - \frac{1}{\lambda} A)$ and $\left\| \frac{1}{\lambda} A \right\| < 1$. $\therefore I - \frac{1}{\lambda} A$ is invertible. So $A-\lambda I$ is invertible. $\therefore \lambda$ is a regular point of A.

**Classification of Spectrum**

1. The point spectrum $\sigma_p(A)$ is the set of Eigen values of A. i.e., $\lambda \in \sigma_p(A)$

$\Leftrightarrow \exists x \in X \setminus \{0\}$ such that AX = $\lambda$X

$\Leftrightarrow \exists x \in X \setminus \{0\}$ such that $(A-\lambda I)x=0$

$\Leftrightarrow \ker (A-\lambda I) \neq \{0\}$

$\Leftrightarrow$ A- $\lambda$I is not on-one.

x is called the eigen vector corresponding to the eigen value $\lambda$.

The dimension of ker (A- $\lambda$I) is called the multiplicity of eigen value $\lambda$

2. The continuous spectrum $\sigma_c(A)$ is defined as follows.

$\lambda \in \sigma_c(A) \Leftrightarrow \lambda \in \sigma(A) \setminus \sigma_p(A)$ and Img(A-$\lambda$I) is dense in X.

3. The residual spectrum $\sigma_r(A) = \left[\sigma(A) \setminus \sigma_p(A) \cup \sigma_c(A)\right]$

So for $\lambda \in \sigma_r(A)$ e have ker(A-$\lambda$I)= $\{0\}$ and Img (A- $\lambda$I) $\neq$ X.

**Remark :** Let $\lambda \notin \sigma_p(A)$ and Img (A-$\lambda$I) = X. ie., ker (A-$\lambda$I) =$\{0\}$ and Img (A-$\lambda$I) = X. then (A-$\lambda$I) : X $\rightarrow$X is one-one and on to. $\therefore$ $\lambda$ is a regular point. So in our classification of $\sigma$ (A), if $\lambda \notin \sigma p(A)$ and $\lambda \in \sigma(A)$ then $\operatorname{Im} g(A-\lambda I) \neq X$.

**Lemma:** Let X be a Banach space and A $\in$ L(X). Let $\lambda_1$, $\lambda_2$, $\lambda_3$ . . . $\lambda_n$ be distinct eigen values of A. Let $Ax_i = \lambda_i x_i$, $x_2 \neq 0$ for i = 1,2 . . . ., n. Then $\{x_1 . . ., x_n\}$ is linearly independent.

**Proof:** of possible let $\{x_1, . . . x_n\}$ be linearly independent.

Let $\alpha_1 x_1 + \alpha_2 x_2 . . + \alpha_n x_n = 0$, wih $\alpha_1 \neq 0$ (1)

Take a polynomial p($\lambda$) with $p(\lambda_1)=1$ and $p(\lambda_2)= p(\lambda_3)=...=p(\lambda_n) =0$. Note that $p(A)x_1=p(\lambda_i)x_i$, for i=1,2 . . . n ie., $p(\lambda_i)$ is an eigen value of the operator p(A). Applying p(A) in (1), we get

$0= \alpha_1 p(A)x_1+\alpha_2 p(A)x_2 + . . . +\alpha_m p(A)\, x_m$

$=\alpha_1 p(\lambda_1)x_1+\alpha_2 p(\lambda_2)x_2 + \ldots + \alpha_m p(\lambda_m)x_m$

$=\alpha_1 1x_1+\alpha_2 0x_2 + \ldots + \alpha_m 0x_m$

$= \alpha_1.x_1.$

This gives $\alpha_1 = 0$, which is a contradiction. Repeat the same procedure for $\alpha_2$, $\alpha_3 \ldots \alpha_n$.

**Note:** Let X be a Banach space. $A \in L(X)$ be an invertible operator. Then $\sigma\left(A^{-1}\right)=\left\{\frac{1}{\lambda} : \lambda \in \sigma(A)\right\}$

**Proof:** Since A is invertible, $0 \notin \sigma(A)$. Now

$$A^{-1}\frac{1}{\lambda}1 = A^{-1}\frac{1}{\lambda}A^{-1}A = \frac{1}{\lambda}A^{-1}(A-\lambda 1)$$

So $(A-\lambda I)$ is not invertible $\Leftrightarrow$ $A^{-1}\frac{1}{\lambda}I$ is invertible

$\therefore$ (A-X1) is not invertible $\Leftrightarrow$ $A^{-1}\frac{1}{\lambda}I$ is not invertible

ie., $\lambda \in \sigma(A) \Leftrightarrow \frac{1}{\lambda} \in \sigma(A^{-1})$

**Example:** Consider the operator $D_w$ in $l_2$ defined by $D_w x) = (w_1x_1, w_2x_2, \ldots)$ for $x = (x_1, x_2, \ldots) \in l_2$. Let us try to find the spectrum of $D_w$.

For $\lambda \in \mathbb{C}$ $(D_w - \lambda I)x = D_w x - \lambda x = (w_1x_1, w_2x_2, \ldots) - (\lambda x_1, \lambda x_2 \ldots)$

$=(w_1-\lambda)x_1, (w_2-\lambda)x_2 \ldots)$

We know that $D_w-\lambda I$ is invertible $\Leftrightarrow \inf_j |w_j-\lambda|>0$

$\Leftrightarrow \lambda \notin \{w_1, w_2 \ldots\}$

$\therefore \sigma(D_w) = \{w_1, w_2\}$

Next let us try to classify the spectrum.

for all j = 1,2,3,. . . we have $D_w e_j = w_j e_j$. $\therefore$ each $w_j$ is an eigen value of $D_w$.

Now let $w_0 \in \{w_1, w_2 \ldots\}$ but $w_0 \neq w_k$ for all k = 1,2,3 . . .

Then $(D_w - w_0 I)x=0 \Leftrightarrow x = 0$

$((w_1 - -w_0)x_1, (w_2 - w_0)x_2, \ldots) = (0,0, \ldots) \Rightarrow$

$x_1 = 0, x_2 = 0, \ldots \Rightarrow x = 0.$

$\therefore D_w - w_0 I$ is one-one and hence $w_0 \notin \sigma_p(D_w)$.

for each j = 1,2,3……, we have

$$ej = \frac{1}{(w_j - w_0)}(w_j - w_0)e_1 = \frac{1}{w_j - w_0}(D_w - W_0 I)e_j \in \text{img}\,(D_w - W_0 I)$$

$$\therefore \text{ So } span\{e_1, e_2 \ldots\} \subset \overline{img[Dw - w_0 I]}$$

i.e. $l_2 \subset img(D_w - w_0 I) \therefore \overline{img(D_w - w_0 I)} = l_2$

This $w_0 \in_c (D_w)$

Remark: consider $T : l_2 \to l_2$ by $Tx = \left[x_1, \frac{x_2}{2}, \frac{x_3}{3}\right]$

Then $\sigma[T] = \left[0,1, \frac{1}{2}, \frac{1}{3}, \ldots\right]$

$$\sigma_p[T] = \left[1, \frac{1}{2}, \frac{1}{3}, \ldots\right], \sigma_c[T] = [0], \sigma r[T] = \phi$$

**Example**: Consider the right shift operator $A: l_2 \to l_2$ by $A(e_{i+1})$

i.e; $A(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots.)$

Clearly A - 0I= A is not one-one and hence not invertible $\therefore 0 \in \sigma(A)$.

**Claim**: If $0 < |\lambda| < 1$ is not one-one and hence not invertible

Proof: We show that $(e_1) = (1, 0, 0, .....) \notin$ Img $(A - \lambda I)$

If possible, let $(A-\lambda I)x=e_1$ for some $x \in l_2$.

i.e; $(-\lambda x_1, x_1 - \lambda x_2, x_2 - \lambda x_3, \ldots) = (1,0,0,\ldots)$

$$\Rightarrow x_1 = -\frac{-1}{\lambda}, x_2 = \frac{-1}{\lambda^2}, x_3 = \frac{-1}{\lambda^3}, \ldots$$

Since $|\lambda| <1$, we have $|x_n| = \frac{1}{|\lambda|^n} \to \infty$ as $n \to \infty$.

$\therefore x = [x_1, x_2, \ldots] \notin l_2$, which is a contradiction. Hence the claim is proved.

Thus we get that $\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subset \sigma(A)$. (1)

Since $\|A\| = 1$ we get that if $|\lambda| > 1$ then $\lambda \in \sigma(A)$.

So $\sigma(A) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. (2)

Since $\sigma(A)$ is closed using (1) and (2), we get that $\sigma(A) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

**claim:** $\sigma_p(A) = \phi$.

**proof:** Clearly $A - 0I = A$ is one-one $\therefore 0 \notin \sigma_p(A)$

Let $0 \neq \lambda \in \mathbb{C}$. Then

$$(A-\lambda I)x = Ax - \lambda x = (0, x_1, x_2, \ldots) - (\lambda x_1, \lambda x_2 \ldots\ldots)$$

$$= (-\lambda x_1, x_1 - \lambda x_2, x_2 - \lambda x_3, \ldots)$$

$$\therefore (A-\lambda I)x = 0 \Rightarrow -\lambda x_1 = 0, x_1 - \lambda x_2 = 0, \ldots$$

$$\Rightarrow x_1 = 0, x_2 = 0, x_3 = 0, \ldots \Rightarrow x = 0.$$

So $A - \lambda I$ is one-one $\therefore \lambda \notin \sigma_p(A)$.

Thus we get $\sigma_p(A) = \phi$

claim: $0 \in \sigma_r(A)$.

proof: We have already obtained $0 \in \sigma_r(A)$.

Also $\overline{img(A-01)} = \overline{imgA} = \left\{(\overline{x_1, x_2 ...}) \in l_2 : x_1 = 0\right\} =$

$\left\{(x_1, x_2, ...) \in l_2 : x_1 = 0\right\} \neq l_2$

$0 \notin \sigma_c(A)$.

Hence $0 \in \sigma_r(A)$.

**Example:** Consider $A : L_2[0,1] \to L_2[0,1]$ defined by $Ax(t) = tx(t)$.

**claim:** $\sigma_p(A) = \phi$

proof: If possible, let $\lambda \in \sigma_p(A)$

Then there exist a non-zero $x \in L_2[0,1]$ such that $Ax = \lambda x$

i.e; $(Ax)(t) = (\lambda x)(t)$, for all $t \in [0,1]$

$tx(t) = \lambda x(t)$, for all t $t \in [0,1]$

$(t - \lambda)x(t) = 0$, for all $t \in [0,1]$

$x(t) = 0$, for all $t \in [0,1]$

i.e. $x = 0$, which is a contradiction.

So $\lambda$ is not an eigen value of A and our claim holds.

**Claim**: If $\lambda \notin [0,1]$, then $\lambda \notin \sigma(A)$

**Proof:** Let $\lambda \notin [0,1]$, consider the function $\frac{1}{t-\lambda}$, which is continuous $[0,1]$.

Now, $\therefore$ the operator $(Bx)(t) = \frac{1}{t-\lambda} x(t)$ is bounded operator in $L_2[0,1]$

Now,

$$(A-\lambda I)x(t) = Ax(t) - \lambda x(t) = tx(t) - \lambda x(t) = (t-\lambda)x(t)$$

So,

$$(A-\lambda I)Bx(t) = (A-\lambda I)\frac{1}{t-\lambda}x(t) = (t-\lambda)\frac{1}{t-\lambda}x(t) = x(t)$$

and

$$B(A-\lambda I)x(t) = B(t-\lambda)x(t) = \frac{1}{t-\lambda}(t-\lambda)x(t) = x(t)$$

So $(A-\lambda I)Bx = B(A-\lambda I)x = Ix$

$\therefore (A-\lambda I)B = B(A-\lambda I) = I$

$\therefore B = (A-\lambda I)^{-1}$. So $(A-\lambda I)$ is invertible.

$\therefore \lambda \notin \sigma(A)$ and our claim holds.

**Claim:** If $\lambda \in [0,1]$, then $\lambda \in \sigma(A)$.

Proof: Let $\lambda \in [0,1]$, Consider the function $f(t) \equiv 1 \in L_2[0,1]$.

$f(t) \notin img(A-\lambda I)$, because if so then we get that there exist $x \in L_2[0,1]$ such that $(t-\lambda)x(t)=1.$

i.e; $x(t)=\frac{1}{t-\lambda} \in L_2[0,1].$

So $(A-\lambda I)$, is not onto and hence not invertible $\therefore$ our claim holds

Thus we get $\sigma_c(A)=[0,1].$

**Lemma:** $\sigma_c(A)=[0,1]$

**Claim(4):** $\sigma_c(A)=[0,1]$

**Proof:** Let $\lambda \in [0,1]$ and $x \in L_2[0,1]$

For each e0, we can consider

$$f(t)=\begin{cases}0, if & t \notin (\lambda-\varepsilon, \lambda+\varepsilon) \\ x(t), if & t \in (\lambda-\varepsilon, \lambda+\varepsilon)\end{cases}$$

Then $(x-f)(t)=\begin{cases}0, if & t \notin (\lambda-\varepsilon, \lambda+\varepsilon) \\ x(t), if & t \in (\lambda-\varepsilon, \lambda+\varepsilon)\end{cases}$

For $e=\frac{1}{n}$ for n = 1, 2, 3, ..... we get $f_n$ such that $\| x-f_n \| \to 0$ as $n \to \infty$

$\therefore f_n \to x$ AS $n \to \infty$. Next we show that $f \in \operatorname{Im} g(A-\lambda I).$

Consider

$$g(t)=\begin{cases}\dfrac{1}{t-\lambda}x(t), if & t\notin(\lambda-\varepsilon,\lambda+\varepsilon)\\ 0, if & t\in(\lambda-\varepsilon,\lambda+\varepsilon)\end{cases}$$

clearly $\in L_2[1,0]$

Note that

$$(A-\lambda I)g(t)=\begin{cases}\dfrac{1}{t-\lambda}x(t), if & t\notin(\lambda-\varepsilon,\lambda+\varepsilon)\\ 0, if & t\in(\lambda-\varepsilon,\lambda+\varepsilon)\end{cases}$$

$$=f(t)$$

So $f\in \operatorname{Im} g(A-\lambda I).$

$\therefore$ We can say that $x\in\overline{img(A-\lambda I)}$

Thus $x\in\overline{img(A-\lambda I)}=L_2[0,1]$, if $\lambda\in[0,1]$

So we get $\sigma_c(A)=[0,1]$

## 5.2 FREDHOLM THEORY OF COMPACT OPERATORS

Let $x$ be infinite dimensional Banach space and $T:X\to X$ be compact operator $T-\lambda I$ and let $\triangle_\lambda=\operatorname{Im}T_\lambda$

**5.2.1 Lemma:** Let $E_1$ be closed subspace of $E$ such that $E\neq E_1\to E\to X$

Then there exist $y_0\in E$ with $\|y_0\|=1$ and such that $dist(y_0,E_1)\geq\dfrac{1}{2}$

**Proof:** Take any $y \in E \setminus E_1$. Let $a = dist(y, E_1)$ since $E_1$ is closed and $y \notin E_1$, we get $a > 0$

choose $x_0 \in E_1$ such that $\| y - x_0 \| < 2a$

consider $y_0 = \dfrac{y - x_0}{\| y - x_0 \|}$. Then $\| y_0 \| = 1$

For every $x \in E_1$, we have

$$\| y_0 - x \| = \left\| \frac{y - x_0}{\| y - x_0 \|} - x \right\|$$

$$= \frac{1}{\| y - x_0 \|} \left\| y - x_0 - \| y - x_0 \| x \right\|$$

$$= \frac{1}{\| y - x_0 \|} \left\| y - \left[ x_0 + \| y - x_0 \| x \right] \right\|$$

$$\geq \frac{1}{2a}, \text{ since } x_0 + \| y - x_0 \| x \in E_1$$

$$= \frac{1}{2}$$

Therefore $dist\left(y_0, E_1\right) \geq \dfrac{1}{2}$

**5.2.2 Corollary:** If $\dim x = \infty$, then the identity operator $I : x \to x$ is not compact

**Proof:** We show that the unit ball $D(x) = \left\{ x : \|x\| \leq 1 \right\}$ is not precompact.

Consider any family of subspaces $E_1 \to E_2 \to E_3 \to E_n \to \ldots\ldots$, where $\dim E_n = n$

They are closed subspaces since they are of finite dimension. By the above lemma, there exist a sequence $y_i \in E_1$ with $\|y_i\| = 1$ such that $dist\left(y_i, E_{(i-1)}\right) \geq \frac{1}{2}$

Then for any $i \neq j, \|y_i - y_j\| \geq \frac{1}{2}$

$\{y_i\}$ is a bounded sequence in $D(x)$, which has no cauchy subsequence. Therefore $D(x)$ is not precompact.

**5.2.3 Proposition:** Let $x$ be infinite dimensional Banach space $T : x \to x$ be a compact operator, Then $0 \in \sigma(T)$

**Proof:** If possible, let $0 \notin \sigma(t)$. Then $T^{-1}$ exists and is bounded.

Therefore $I = T^{-1}T$ is compact (since the compact operators form an ideal) which is a contradiction. Hence $0 \in \sigma(t)$

**5.2.4 Proposition:** $\forall \varepsilon > 0$, there is only a finite number of linearly independent eigen vectors corresponding to eigen values $\lambda_i$ with $|\lambda_i| \geq \varepsilon$

That is $\exists$ a finite number of $\lambda_i \in \sigma_p$ such that $|\lambda_i| \geq \varepsilon$ and every $\lambda_i$ has finite multiplicity, $\dim(\ker T_{\lambda i}) < \infty$

**Proof:** If possible, let there be linearly independent vectors $x_1, x_2, .....x_k$ with $Tx_i = \lambda_i x_i$ for $i = 1, 2.....|\lambda_i| \geq \varepsilon$.

Consider $E_k = span\{x_1, x_2, .....x_k\}$ for $k = 1, 2, ...$

Then clearly $E_k$ is a closed subspace of $E_{k+1}$ and $E_k \neq E_{k+1}$. Therefore by the lemma we proved earlier, $\exists y_k \in E_k$ with $\|y_k\| = 1$ and $dist(y_k, E_{k-1}) \geq \frac{1}{2}$

Claim: $\left\{\frac{y_k}{\lambda_k}\right\}_{k=2}^{\infty}$ is bounded sequence, where as $\left\{T\frac{y_k}{\lambda_k}\right\}_{K=2}^{\infty}$ does not have a cauchy subsequence.

proof: $\left\|\frac{y_k}{\lambda_k}\right\| = \frac{1}{|\lambda_k|}\|y_k\| \leq \frac{1}{\varepsilon} \qquad \forall k = 1, 2, ...$

Therefore $\left\{\frac{y_k}{\lambda_k}\right\}_{k=2}^{\infty}$ is bounded sequence.

Now let $y_k = a_1 x_1 + ... + a_k x_k$.

Then $\frac{y_k}{\lambda_k} = \frac{a_1}{\lambda_k} T(x_1) + ... + \frac{a_k}{\lambda_k} T(x_k)$

$= \frac{\overline{a_1 \lambda_1}}{\lambda_k} x_1 + .... + \frac{a_{k-1}\lambda_{k-1}}{\lambda_k} x_{k-1} + a_k x_k$

$= a_k x_k + a_{k-1} x_{k-1} + ... + a_1 x_1 + \left(\frac{a_1 \lambda_1}{\lambda_k} - a_1\right) x_1 + \left(\frac{a_2 \lambda_2}{\lambda_k} - a_2\right) x_2 + ... +$

$\left(\frac{a_{k-1}\lambda_{k-1}}{\lambda_k} - a_{k-1}\right) x_{k-1}$

$= y_k + z_k$, for some $z_k \in E_{k-1}$

Then for any $k > n$, we have

$$\left\| T\left(\frac{y_k}{\lambda_k}\right) - T\left(\frac{y_n}{\lambda_n}\right) \right\| = \left\| y_k + z_k - (y_n + z_n) \right\|$$

$$= \left\| y_k - (y_n - z_k + z_n) \right\|$$

$$\geq \frac{1}{2}, \text{ since } y_n - z_k + z_n \in E_{k-1}$$

Therefore the claim is proved.

This shows that T is not compact, which is a contradiction.

**Remarks:** The point spectrum $\sigma_p(t)$ of a compact operator is a countable set having 0 as the only possible limit point. Every eigen space of T corresponding to a non zero eigen value of T is finite dimensional.

**5.2.5. Proposition:** Let T be a compact operator and $\lambda \neq 0$. Then $\Delta_\lambda = \overline{\Delta_\lambda}$. That is $\Delta_\lambda$ is a closed subspace.

**Proof:** Let $y \in \Delta_\lambda$. Then $\exists x \in X$ such that $T_\lambda x = y$.

Let $E_y = \{z \in x; T_\lambda z = y\}$ and $E_0 = \ker T_\lambda$

Then clearly i $E_y = x + E_0$

Let $\alpha(y) = \inf \{\|z\|, z \in E_y\}$

**Claim:** $\exists$ a constant C independent of $y$ such that $\alpha(y) \leq C\|y\|$.

Proof: Assume the contrary that is, $\exists$ a sequence $\{\bar{y}_n\}$ s.t. $\dfrac{\|\bar{y}_n\|}{\alpha(\bar{y}_n)} \to 0$

Take $\overline{y_n}$ such that $\frac{\|\bar{y}_1\|}{\alpha(\bar{y}_1)} < 1, \bar{y}_2$ such that $\frac{\|\bar{y}_2\|}{\alpha(y_2)} < \frac{1}{2}$ and so on

Generally take $\bar{y}_n s.t. \frac{\|\bar{y}_n\|}{\alpha(\bar{y}_n)} < \frac{1}{n} \to 0$ as $n \to \infty$

Since $T_\lambda x = y$ and only if $T_\lambda(rx) = ry$ for every scalar r, the function $\alpha(y)$ is homogeneous i.e. $\alpha(ry) = r\alpha(y)$ for $r \to 0$

Let $r > 0$ then,

$\alpha(ry) = \inf\{\|v\| : T_\lambda v = ry\}$

$= \inf\{\|rz\| : T_\lambda(rz) = ry\}$

$= \inf\{\|rz\| : T_\lambda z = y\} = |r| \inf\{\|z\|, T_\lambda z = y\}$

$= r\alpha(y).$

For $r = 0$ clearly $\alpha(ry) = 0$ and $r\alpha(y) = 0$

Take $y_n = \frac{\bar{y}_n}{\alpha(\bar{y}_n)}$. Then $\alpha(y_n) = 1$

Also $\|yn - 0\| = \|y_n\| = \frac{\|\bar{y}_n\|}{\alpha(\bar{y}_n)} \to 0$ as $n \to \infty$

Choose $x_n s.t. T_\lambda x_n = y_n$ and $\|x_n\| < 2$

(since $\alpha(y_n) = 1$ we have $inf\{\|z_n\|; T_\lambda z_n = y_n\} = 1$

Therefore we can choose $x_n$ such that $T_\lambda x_n = y_n$ and $\|x_n\|$ is close to 1.

i.e., $\exists x_n$ such that $T_\lambda x_n = y_n$ and $\|x_n\| \leq 2$)

Since T is compact, $\exists$ a subsequence $x_{nk}$ such that $Tx_{nk} \to w$

Also $Tx_{nk} - \lambda x_{nk} = (T - \lambda I)x_{nk} = T_\lambda x_{nk} = y_{nk} \to 0$.

Therefore $\|\lambda x_{nk} - w\| = \|\lambda x_{nk} - Tx_{nk} + Tx_{nk} - w\| \leq \|\lambda x_{nk} - Tx_{nk}\| + \|Tx_{nk} - w\| \to 0$.

So $\lambda x_{nk} \to w$ So $x_{nk} \to w/\lambda$. Let $w/\lambda = x_0$.

Thus, we have $T_\lambda(x_0) = T_\lambda(\lim x_{nk} = \lim T_\lambda(x_{nk}) = \lim y_{nk} = 0$.

Therefore $x_0 \in E_0$.

Then clearly $x_{nk} - x_0 \in \in E_{ynk}$ ($T_\lambda(x_{nk} - x_0) = T_\lambda x_{nk} - T_\lambda x_0 = y_{nk} - 0 = y_{nk}$).

Therefore $\alpha(y_{nk}) \to 0$ ($\alpha(y_{nk} = \inf\{\|z=_{nk}\|;\ T_\lambda z_{nk} = y_{nk}\} = \inf\{\|z_{nk}\|;\ z_{nk} \in E_{ynk}\}$

$\leq \|x_{nk} - x_0\|$ since $x_{nk} - x_0 \in E_{ynk}$

$\to 0$ since $x_{nk} \to x_0$)

which is a contradiction.

Hence the claim is proved.

Now let, $y_n \in \Delta_\lambda$ and $y_n \to y$.

Note that $\{y_n\}$ is bounded (since every covergent sequence is bounded)

By the above claim $\exists$ a constant c such that $\alpha(y_n) \leq c\|y_n\| \forall_n = 1, 2, 3, .....$

So $\alpha(y_n) < k$, for some constant k. (since $(y_n)$ is bounded)

Then $\exists x_n$ with $\|x_n\| \leq k$ and $T_\lambda x_n = y_n$.

If possible, let $\|x_n\| > k$ for all $x_n$ with $T_\lambda x_n = y_n$.

Then we get that $\inf\{\|x_n\|;\ T_\lambda x_n = y_n\} \geq k$

i.e., $\alpha(y_n) \geq k$, which is a contradiction.)

We note that $T_\lambda x_n \to y$. (since $y_n \to y$ and $y_n = T_\lambda x_n$)

Since $\{x_n\}$ is bounded and T is compact, $\exists$ a sub sequence $x_{nk}$ s.t. $Tx_{nk} \to w$.

Therefore we get that $x_{nk} \to \dfrac{w-y}{\lambda} = x_0$ (We have $T_\lambda x_{nk} \to y$

i.e., $Tx_{nk} - \lambda x_{nk} \to y$

Also we have $Tx_{nk} \to w$

Therefore $\lambda x_{nk} = Tx_{nk} - (Tx_{nk} - \lambda x_{nk}) \to w - y$

Then $x_{nk} \to \dfrac{w-y}{\lambda}$ )

Hence $T_\lambda x_0 = y$ (we have $x_{nk} \to x_0$ and $T_\lambda x_{nk} \to y$. But since $T_\lambda$ is bounded

$T_\lambda x_{nk} \to T_\lambda x_0$

Therefore $T_\lambda x_0 = y$

Thus $y \in \Delta_\lambda$

Hence $\Delta_\lambda$ is a closed subspace.

**5.2.6. Corollary:**

Let T be a compact operator and T* be its dual operator. $\Delta\bar{\lambda}^*$ = $\mathrm{im}T\bar{\lambda}^* \to X$.

Then $\Delta\bar{\lambda}^* = \bar{\Delta}\ \bar{\lambda}^*$ for any $\lambda \neq 0$.

**Proof:** Since T is compact, T* is also compact.

Therefore the corollary is obvious from the above proposition. (In the proposition, replace T with T*, $\lambda$ with $\bar{\lambda}$ and $\Delta_\lambda$ with $\Delta\bar{\lambda}^*$.

**5.2.7. Proposition**

X be a Banach space. T a compact operator on X. $\lambda \neq 0$

(1) If $\Delta\lambda = X$ then $\ker T_\lambda = \{0\}$

(2) If $\Delta\overline{\lambda}^* = X^*$ then $\ker T\overline{\lambda}^* = \{0\}$

**Proof**: (1) Let $\Delta_\lambda = x$. If possible let there be some non zero $x_0 \in \ker T_\lambda$

i.e., $T_\lambda x_0 = 0$

Since $\Delta_\lambda = \text{im} T_\lambda = x$,

$\exists x_1$ such that $T_\lambda x_1 = x_0$

Similarly $\exists x_2$ such that $T_\lambda x_2 = x_1$

In general, $\exists x_k$ such that $T_\lambda x_k = x_{k-1}$, for k = 1, 2, 3, ....

Let $N_k = \{x: T^k_\lambda x = 0\} = \ker T^k_\lambda$ for k = 1, 2, 3, ....

CLAIM: $N_k$ is a proper closed subspace of $N_{k+1}$ for k = 1, 2, 3, ...

Prf: Since $T^k_\lambda$ is continuous, we get that $N_k$ is closed. Let $x \in N_k$. Then $T^k_\lambda x = 0$.

Therefore $T_\lambda{}^{k+1} x = T_\lambda(T^k_\lambda x) = T_\lambda(0) = 0$.

So $x \in N_{k+1}$ therefore $N_k \subset N_{k+1}$

We have $T^k_\lambda x_k = T_\lambda{}^{k-1}(T_\lambda x_k) = T_\lambda{}^{k-1}(x_{k-1}) = ..... = T_\lambda(x_1) = x_0 \neq 0$. Therefore $x_k \neq N_k$.

But $T_\lambda^{k+1} x_k = T_\lambda(T_\lambda^k x_k) = T_\lambda(x_0) = 0$

Therefore $x_k \in N_{k+1}$

Hence our claim is proved.

Therefore by the lemma, we proved earlier, $\exists y_{k+1} \in N_{k+1}$ such that $\|y_{k+1}\| = 1$ and $\text{dist}(y_{k+1}, N_k) \geq ½$

Note that $\{y_k\}$ is a bounded sequence.

We show that $\{Ty_k\}$ does not have a cauchy subsequence.

Let $k > n$.

Then $\|Ty_k - Ty_n\| = \|Ty_k - \lambda y_k + \lambda y_k - Ty_n + \lambda y_n - \lambda y_n\|$

$= \|T_\lambda y_k + \lambda y_k - T_\lambda y_n - \lambda y_n\|$

$= \|\lambda y_k - (\lambda y_n + T_\lambda y_n - T_\lambda y_k)\|$

$= |\lambda| \, \|y_k - (y_n + T_\lambda\left(\frac{y_n}{\lambda}\right) - T_\lambda\left(\frac{y_k}{\lambda}\right)\|$

$\geq |\lambda|$½, since $y_n + T_\lambda\left(\frac{y_n}{\lambda}\right) - T_\lambda\left(\frac{y_k}{\lambda}\right) \in N_{k-1}$ and

$dist(y_k, N_{k-1}) \geq$ ½

(Let $z = y_n + T_\lambda\left(\frac{y_n}{\lambda}\right) - T_\lambda\left(\frac{y_k}{\lambda}\right)$ Then

$$T_\lambda^{k-1}(z) = T_\lambda^{k-1}(y_n) + T_\lambda^k\left(\frac{y_n}{\lambda}\right) - T_\lambda^k\left(\frac{y_k}{\lambda}\right)$$

We have $y_n \in N_n \subseteq N_{k-1}$ Therefore $T_\lambda^{k-1}(y_n) = 0$

$y_n \in N_n \subseteq N_{k-1}$ Therefore $T_\lambda^k\left(\frac{y_n}{\lambda}\right) = \frac{1}{\lambda} T_\lambda^k(y_n) = 0$

$y_k \in N_k$ Therefore $T_\lambda^k\left(\frac{y_k}{\lambda}\right) = \frac{1}{\lambda} T_\lambda^k(y_k) = 0$

$\therefore$ $\{Ty_k\}$ has no Cauchy Sequence.

so T is not compact, which is a contradiction.

Hence we must have $\ker T_\lambda = \{0\}$.

(ii) Follows from (i) Since T* is also compact operator.

**Recall**

HB Theorem: X-normed Space. E subspace of X. $f_o \in E^*$ Then $\exists$ an extension $f \in X^*$ such that $f|_E = f_0$ and $\| f \| = \| f_0 \|$.

**corollary**: Let L $\rightarrow$ X be a subspace of a normed space X and let x $\in$ X be such that dist(x, L) = d > 0. The $\exists f \in$ X* such that $\| f \| = 1$, $f(L) = 0$ and $f(x) = d$.

**Note:** For any L $\rightarrow$ X, the annihilator $L^{\perp} = \{ f \in X^*, f(L) = 0\}$

$L^{\perp}$ is a closed subspace of X*. Clearly $L^{\perp} = \overline{L}^{\perp}$. For $F \rightarrow X^*$, $F_{\perp} = \bigcap_{f \in F} \ker f$

**corollary**: Let $L$ is a closed subspace. Then $(L^{\perp})_{\perp} = L$.

**Note:** Let $A \in L(X,Y)$. Then $(ImA)^{\perp} = ker\ A^*$ and $kerA = (ImA^*)_{\perp}$.

**Note**: (A+B)* = A* + B*

we agree to write $(\lambda I)^* = \overline{\lambda} I$ .

**5.2.8. Proposition:** X-Banach Space. T-Compact Operator. $\lambda \neq 0$

i. (a) $\Delta_{\lambda}^{\perp} = \ker T_{\bar{\lambda}}^*$

(b) $\Delta_{\lambda} = \neq$ X implies $\ker T_{\bar{\lambda}}^* \neq \{0\}$

(c) $\Delta_{\lambda} = (\ker T_{\bar{\lambda}}^*)_{\perp}$

ii. (a) $\ker T_{\lambda} = \left(\Delta_{\bar{\lambda}}^*\right)$; and

(b) $(\ker T_{\lambda})^{\perp} = \left(\Delta_{\bar{\lambda}}^*\right)$

**Proof (i):**

(a) We have $[Im(T-\lambda I]^{\perp} = \ker(T-\lambda I)^*$ since $(ImA)^{\perp} = ImA^*$

ie; $\Delta_\lambda^\perp = \ker(T^* - \bar{\lambda} I) = \ker T_{\bar{\lambda}}^*$.

(b) Assume that $\Delta_\lambda \neq X$. Since $\Delta_\lambda$ is closed $\bar{\Delta}_\lambda = \Delta_\lambda$
$\therefore \exists x \in X$ such that dist $(x, \Delta_\lambda) = d > 0$.

Then by the corollary of the HB theorem,

$\exists f \in X^*$ such that $f(\Delta_\lambda) = 0$ and $f(x) = d \neq 0$

So $f \in (\Delta_\lambda)^\perp$ and $f \neq 0$

i.e., $f \in \ker(T_{\bar{\lambda}})^*$ and $f \neq \{0\}$, by (a)

$\therefore \ker(T_{\bar{\lambda}}^* \neq \{0\}$.

(c) Since $\Delta_\lambda$ is a closed subspace, we have $(\Delta_\lambda^\perp)_\perp = \Delta_\lambda$. i.e., $(\ker T_{\bar{\lambda}}^*)_\perp = \Delta_\lambda$.

**Proof (ii):**

(a) we have $\ker(T - \lambda I) = [\mathrm{Im}(T - \lambda I)^*]_\perp$, since $\ker A = (\mathrm{Im} A^*)$. i.e., $\ker T_\lambda = [\mathrm{Im}(T^* - \bar{\lambda} I)_\perp = \left(\Delta_{\bar{\lambda}}^*\right)_\perp$.

(b) Assume that $f \in \Delta \overset{*}{\lambda}$.

Then $\exists h \in X^*$ such that $T \overset{*}{\lambda} h = f$ $\therefore$ for any $x \in \ker T\lambda$, we have

$$f(x) = (T \overset{*}{\lambda})(x) = h(T_\lambda x) = h(0) = 0$$

[since

$$(T\lambda)^* = (T - \lambda I)^* = T^* - \lambda I = T \overset{*}{\lambda}; A^* f(x) = f(Ax)]$$

So $f \in \left(\ker T_\lambda\right)^\perp$

$\therefore \Delta \overset{*}{\lambda} \subseteq \left(\ker T \lambda^{\perp}\right)$

Next assume that $f \in (\ker T\lambda)^{\perp}$ i.e; $f(x)=0 \ \forall x \in \ker T\lambda$

We try to show that $\exists h \in X^*$ such that $T \overline{\lambda}^{*} h = f$

We define a linear functional $h_0$ on $\Delta_\lambda$ as follows;

For $y \in \Delta \lambda$, define ($h_0(y) = f(x)$, where x is such that $T_\lambda x = y$.$h_0$ is well that defined because if $x_1$ is such that $T_\lambda x_1 = y$, then $x-x_1$ is well$\in \ker T_\lambda$ and hence $f(x - x_1) = 0$ which gives $f(x) = f(x_1)$.

Thus $\exists$ a constant C such that $\alpha(y) = \inf \{||x||; T_\lambda x = y\} \leq C||y||$. [ by the claim of 5.2.5 proposition]

$\therefore$ we can take x such that $||x|| \leq (C+1)\ ||y||$

Thus, we obtain $|h_0(y)| = |f(x)| \leq ||f||\ ||x|| \leq ||f||(C+1)\ ||y||$.

so $h_0$ is a bounded linear functional on $\Delta\lambda$.

$\therefore$ by HB theorem an extension $h \in X^*$ of $h_0$ on all the space. Hence for any $x \in X$, we have

$$\left(T \overset{*}{\lambda} h\right)(x) = h(T_\lambda x) = h_0(T_\lambda x) = f(x).$$

This gives $T \overset{*}{\lambda} h = fie; f \in \Delta \overset{*}{\lambda}$

So we get $(\ker T_\lambda)^{\perp} \subseteq \Delta \overset{*}{\lambda}$

Hence $\left(\ker T_\lambda\right)^{\perp} = \Delta \overset{*}{\lambda}$

Fredholm's First Theorem

5.2.9 Proposition: X-Banach Space. T-Compact Operator. $\lambda \neq 0$. Then $\ker T^{*}_{\lambda} \neq \{0\}$ if and only if $\ker T_{\lambda} \neq \{0\}$.

Proof: $\ker T^{*}_{\lambda} \neq \{0\} \Rightarrow \Delta^{*}_{\lambda} \neq X^{*}$ *by* 5.2.7(*ii*)

$\Rightarrow (\ker T_{\lambda})^{\perp} \neq X^{*}$ by 5.2.8 (ii) (b)

$\Rightarrow \ker T_{\lambda} \neq \{0\}$

conversely

$\ker T_{\lambda} \neq \{0\} \Rightarrow \Delta_{\lambda} \neq X$ by 5.2.7(i)

$\Rightarrow \ker T^{*}_{\lambda} \neq \{0\}$ *by* 5.2.8(*i*)(*b*)

**5.2.10. Proposition:** X-Banach Space T-Compact Operator $\lambda \neq \{0\}$.

$\ker T_{\lambda} = \{0\} \Rightarrow \Delta_{\lambda} = X$.

**Proof**: $\ker T_{\lambda} = \{0\} \Rightarrow \ker T^{*}_{\lambda} = \{10\} \Rightarrow \Delta\lambda = X$ by 5.2.8(i) (b)

**5.2.11 Theorem:** For any compact operator T: $X \to X$ in an infinite dimensional Banach Space, $\sigma(T) = \{0\} \cup \sigma_p(T)$

[Proof: By 5.2.3., $0 \in \sigma(T)$. For $\lambda \neq 0$ by. 5.2.7.(i) and 5.2.10 $T_{\lambda}$ is one-one iff $T_{\lambda}$ is onto.

So if $0 \neq \lambda \notin \sigma_p(T)$, the $T_{\lambda}$ is one-one and onto, and hence by bounded inverse theorem $T_{\lambda}$ is invertible $\therefore \lambda \notin \sigma(T)$. Thus $\sigma(T) = (T) = \{0\} \cup \sigma_p(T)$.

**Note:** 5.2.9 is known as Fredholm's first theorem. it states that for $\lambda \neq 0$, $T_{\lambda}$ is not one-one $\Leftrightarrow T^{*}_{\lambda}$ is not one-one i.e; for $\lambda \neq 0$,

$\lambda \notin \sigma_p(T) \Leftrightarrow \overline{\lambda} \notin \sigma_p(T^*)$.

$\therefore$ we have $\sigma p(T) - \{0\} = \overline{\sigma_p(T^*)} - \{0\}$

**Example: The Volterra Operator:** Let X = C[0.1] and let k(t, s) be a continuous function. The operator T: X$\rightarrow$X given by T(x) (t) = $\int_0^t k(t,s)x(s)ds = y(t)$ is known as the volterra operator.

Claim: The equation $x(t) - T(x)(t) = x(t) - \int_0^t k(t,s)x(s)ds = y(t)$

has a solution for every y$\in$X.

i.e., T-I is onto.

Proof: Let max $_{t,\,s,}$ |k(t, s)| = C and max$_t$ |y(t)|= $C_1$

Let $x_0(t) = 0$ and $x_n(t) = y(t) + \int_0^t k(t,s)x_{n-1}(s)ds$ for n = 1,2..(1)

We show that $|x_{n+1}(t) - x_n(t)| \leq \frac{c_1 c^n t^n}{n!}$ for n = 0,1,2.....(2)

For n = 0 we have $|x_1(t) - x_0(t)| = |x_1(t)| = |y(t)| \leq c_1$, so the result (2) holds for n = 0.

As induction hypothesis, we assume $|x_n(t) - x_{n-1}(t) \leq \frac{c_1 c^{n-1} t^{n-1}}{(n-1)!}$

Now, $|x_{n+1}(t) - x_n(t)| \leq \int_0^t |k(t,s)|\,|x_n(t) - x_{n-1}(t)|\,ds \leq$

$$C\int_0^t \frac{C_1 C^{n-1} t^{n-1}}{n-1)} = C_1 C^n \frac{t^n}{n!}$$

So (2) is proved by mathematical induction

Note that

$x_n(t) = \sum_{k=1}^{n} [x_k(t) - x_{k-1}(t)]$ and

$$| x_k(t) - x_{k-1}(t) | \leq C_1 \sum_{k=1}^{\infty} \frac{c^k t^k}{k!} = C_1(e^{ct} - 1)$$

i.e., $\sum_{k=1}^{\infty} [x_k(t) - x_{k-1}(t)]$ converges absolutely.

Therefore $\sum_{k=1}^{\infty} [x_k(t) - x_{k-1}(t)]$ converges uniformly.

Thus, we get that $(x_n)$ is a convergent sequence in X. So $x_n \to s$ in X.

Therefore, we get from (1), that $x(t) = y(t) + \int_0^t k(t,s)x(s)ds$

Hence, the claim is proved.

By 5.2.7 (1) T-x is one-one. (Note that T is a complete operator)

$\therefore 1 \notin \sigma_p(t)$.

By taking kernal function as $\lambda k(t,s)$, we can prove that $\lambda \notin \sigma_p$ (t), for any other $\lambda \neq 0$

Hence we get that $\sigma(t) = \{0\}$

(it is clear that T is not one-one since for $x(t) = \begin{cases} 1 & if\ t = 1/2 \\ 0 & o.w. \end{cases}, T(x)(t)_ = 0$

$\therefore 0 \in \sigma_p(T)$

**Problem 1**: For what $g \in c[0, II]$ does the integral equation

$$f(x) - \int_0^{\pi} \sin(x+y) f(y) dy = g(x)$$

has a solution in $c[0, \pi]$

**Solution:** The given equation can be written as

$f(x) - (Tf)(x) = g(x)$

Where,

$$\left( Tf(x) = \int_0^{\pi} \sin(x+y) f(y) dy = \sin x \int_0^{\pi} \cos y f(y) dy + \cos x \int_0^{\pi} \sin f(y) dy \right)$$

Note that T is a compact operator.

Let us consider the corresponding homogenous equation

$f(x- (Tf)(x) = 0$

Then we get $f(x) = (Tf)(x) = a\sin x + b\cos x$

where, $a = \int_0^{\pi} f(x) \cos x dx \;\&\; b = \int_0^{\pi} f(x) \sin x dx$

Now, $f(x) \cos x = a\sin x\cos x + b\cos^2 x$

$$\therefore a = a\int_0^{\pi} \sin x \cos x + b\int_0^{\pi} \cos^2 x dx = \frac{b\pi}{2}$$

Also, $f(x) \sin x = a \sin^2 x + b\sin x\cos x$

$$\therefore b = b\int_0^{\pi} \sin x \cos x dx + a\int_0^{\pi} \sin^x dx = \frac{a\pi}{2}$$

Hence we get, $\frac{b\pi}{2} . \frac{\pi}{2} = \frac{b\pi^2}{4} \Rightarrow b = 0, \therefore a = 0.$

So the equation $f(x) - (Tf)(x) = 0$ has only one solution $f(x) = 0$

Thus we get that T-I is one-one $\therefore$ T-I is onto by 5.2.10

So the equation $f - Tf = g$.

**Problem 2:** Define the operator

$$k : L_2[0,1] \to L_2[0,1] by kf(t) = \int_0^1 k(t,s) f(s) ds$$

where $k(t,s)\begin{cases}1, s \le t\\ 0, s \ge t\end{cases}$

Find the solution of k.

**Solution:** Since k is a compact operator, $0 \in \sigma(k)$

Let $\lambda \neq 0$ be such that $kf(t) = \lambda f(t)$

i.e., $\int_0^1 k(t,s) f(s) ds = \lambda f(t)$

i.e. $\int_0^t f(s) ds = \lambda f(t)$

By fundamental theorem of calculus, we get that $\lambda f 1 = f$. Also we get that $f(0) = 0$

solving the initial value problem $\lambda f^{1} = f, f(0) = 0$, we get that $f = 0$

$\therefore \lambda \neq 0$ cannot be an eigen value of k.
Thus we obtain $\sigma(k) = \{0\}$.

clearly k is one-one.

$\therefore$ 0 is not an eigen value of k. So, $\sigma p(k) = \Phi$

## 6.1 Self Adjoint Operators

Let H be a Hilbert space. A bounded operator $A : H \to H$ is called self adjoint or symmetric if and only if $(Axy) = x,Ay)$ for all $x, y \in H$.

### 6.1 General Properties:

**Property 6.1.2.:** The point spectrum $\sigma_p$ of a self adjoint operator A satisfied $\sigma_p(A) \subseteq \mathbb{R}$ .

**Proof :** Let $\lambda \in \sigma_p(A)$

Then $\exists\ x \neq 0$ such that $Ax = \lambda x$

We get

$$\lambda\|x\|^2 = \lambda\langle x,x\rangle = \langle \lambda x,x\rangle = \langle Ax,x\rangle$$
$$= \langle x,Ax\rangle = \langle x,\lambda x\rangle = \bar{\lambda}\langle x,x\rangle = \bar{\lambda}\|x\|^2$$

This gives $\lambda = \bar{\lambda}$

$\therefore \lambda \in \mathbb{R}$

Thus $\sigma_p \subseteq \mathbb{R}$

**Property 6.1.2:** If $\lambda_1, \lambda_2 \in \sigma_p(A)$, $\lambda_1 \neq \lambda_2$ and $Ax_1 = \lambda_1 x_1$, $Ax_2 = \lambda_2 x_2$, then $x_1 \perp x_2$. (Here A is a symmetric operator)

**Proof:** $\lambda 1\langle x_1,x_2\rangle = \langle \lambda_1 x_1,x_2\rangle = \langle Ax_1 x_2\rangle = \langle x_1,Ax_2\rangle$

$$= \langle x_{1=,}\lambda_1 x_2\rangle, = \bar{\lambda}\langle x_1,x_2\rangle = \lambda_2\langle x_1,x_2\rangle$$

i.e; $(\lambda_1 - \lambda_2)\langle x_1,x_2\rangle = 0$

Since $\lambda_1 \neq \lambda_2$ we get $\langle x_1, x_2\rangle = 0$

$\therefore x_1 \perp x_2$

A subspace L of H is called is invariant with respect to an operator A if and only if $A(L) \subseteq L$.

**Property 6.1.3:** If L is invariant with respect to a symmetric operator. A then so is $L^{\perp}$

**Proof:** Consider $x \in L$ and $y \in L^{\perp}$

Since L is invariant with respect to A, we have $Ax \in L$.

$\therefore \langle Ax, y\rangle = 0$

So $\langle x, Ay\rangle = 0$ since A is symmetric.

$\therefore$ we get that $Ay \in L^{\perp}$

Thus $L^{\perp}$ invariant with respect to A.

**Property 6.1.4:** If A and B are symmetric operators and AB = BA, then AB is also symmetric.

**Proof:** We have

$\langle (AB)x,y\rangle = \langle A(Bx),y\rangle = \langle Bx, Ay\rangle$

$\langle x,B(Ay)\rangle = \langle x,(BA)y\rangle = \langle x,(AB)y\rangle$

$\therefore$ AB is symmetric

**Property 6.1.5:** If A is a symmetric operator, then

$$\|A\| = \sup_{x\neq 0} \frac{|\langle Ax, x\rangle|}{\|x\|^2}$$

**Proof:** Let C = $\sup_{x\neq 0} \frac{|\langle Ax, x\rangle|}{\|x\|^2}$

By Cauchy-Schwartz inequality, we have

$$|(Ax, x)| \leq \|Ax\|\,\|x\| \geq \|A\|\|x\|^2, \forall x \in H$$

$$\therefore \frac{|\langle Ax, x\rangle|}{\|x\|^2} \leq \|A\| \forall x \neq 0$$

So $\sup_{x\neq 0} \frac{|\langle Ax, x\rangle|}{\|x\|^2} \leq \|A\|$

i.e., C$\leq$ ||A||

Next, we note that $\langle A(x+y), x+y\rangle - \langle A(x-y), x-y\rangle = 2(\langle Ax,y\rangle + \langle Ay,x\rangle)$

[LHS = $\langle Ax,x\rangle + \langle Ax,y\rangle + \langle Ay,x\rangle + \langle Ay,y\rangle - [\langle Ax,x\rangle - \langle Ax,y\rangle - \langle Ay,x\rangle + \langle Ay,y\rangle]$

= $2\langle Ax,y\rangle + 2\langle Ay,x\rangle$ = RHS]

$$\therefore |\langle Ax, y\rangle + \langle Ax, y\rangle| \leq \frac{1}{2}\left\{|\langle A(x+y), x+y\rangle| + |\langle A(x-y), x-y\rangle|\right\}$$

by triangle inequality

i.e; $\left|\langle Ax, y\rangle + \langle y, Ax\rangle\right| \le \frac{1}{2}\left\{C\|X+y\|^2 + C\|x-y\|^2\right\}$

$= \frac{C}{2}\left\{2\|x\|^2 + 2\|y\|^2\right\}$ by parallelogram law

$= C\,(\|x\|^2 + \|y\|^2)$ (*)

Claim: For all $x \in H$ with $\|x\| = 1$, we have $\|Ax\| \le C$.

Proof: If $Ax = 0$, for $x \in H$ with $\|x\| = 1$ them clearly the claim holds

So let $x \in H$ with $\|x\| = 1$, but $Ax \neq 0$.

Consider $y = \frac{Ax}{\|Ax\|}$. So $\|y\| = 1$

Then by (*) we have

$$\left|\left\langle Ax, \frac{Ax}{\|Ax\|}\right\rangle + \left\langle \frac{Ax}{\|Ax\|}, Ax\right\rangle\right| \le C(1+1)$$

i.e; $\frac{1}{\|Ax\|}\left|\ \langle Ax, Ax\rangle + \langle Ax, Ax\rangle\right| \le 2C$

i.e; $2\frac{\|Ax\|^2}{\|Ax\|} \le 2C$

i.e; $\|Ax\| \le C$

Hence the claim is proved.

$\therefore$ we get

$\sup_{\|x\|=1} \|Ax\| \le C$

i.e; $\|Ax\| \le C$

Hence we get $\|A\| = C$

**Property 6.1.6** $\langle Ax, x\rangle \in \mathbb{R}$ for every $x \in H$ iff A is symmetric.

**Proof:** Let A be symmetric. Them we have, $\forall\ x \in H$

$$\langle Ax, x\rangle = \langle x, Ax\rangle = \overline{\langle Ax, x\rangle}$$

$\therefore \langle Ax, x\rangle \in \mathbb{R}$

Conversely, assume that $\langle Ax, x\rangle \in \mathbb{R}\ \forall x \in H$

We not that

$4\langle Ax, y\rangle = \langle A(x+y), x + y\rangle - \langle A (x-y), x-y\rangle + i \langle A(x+iy), x+iy\rangle - i\langle A(x-iy), x-iy\rangle$

[We have $\langle A(x+y), x+y\rangle - \langle A(x-y), x - y\rangle + 2\langle Ax, y) + \langle 2Aiy, x\rangle$

Also $\langle A(x+iy), x + iy\rangle - \langle A(x-iy), x-iy\rangle = 2\langle Ax, iy\rangle + 2 \langle A(iy), x\rangle$

$\therefore$ RHS $= 2 (Ax, y) + 2\langle Ay, x \rangle + 2i\langle Ax, iy \rangle + 2i \langle A (iy), x \rangle$

$= 2 (Ax, y) + 2\langle Ay, x \rangle + 2\langle Ax, y \rangle - 2 \langle Ay, x \rangle$

$= 4 \langle Ax, y\rangle]$

By interchanging the positions of x and y, we get

$4 \langle Ay, x\rangle = \langle A(y+x\rangle, y+x\rangle - \langle A(y-x) y-x\rangle + i\langle A(y+ix), y+ix\rangle - i \langle A (y-ix), y-ix\rangle$

Taking conjugates on both sides and use the fact that $\langle Az, z\rangle \in \mathbb{R}$, for any $z \in H$, we get

$4 \langle x, Ay\rangle = \langle A(y+x), y+c\rangle - \langle A(y-x), y-x - i \langle A (y+ix), y+ix\rangle + i\langle A(y-ix), y-ix\rangle$

$= \langle A(x+y), x+y\rangle - \langle A(x-y), x-y\rangle - i\langle iA(x-iy), i(x-iy)\rangle + i\langle - iA (x+iy), -i(x+iy)\rangle$

$= \langle A(x+y), x+y\rangle - \langle A (x-y), x - y\rangle - i \langle A(x-iy), x- iy\rangle + \langle iA(x+iy), x+iy\rangle$

$= 4\langle Ax, y\rangle$

$\therefore$ we get $\langle Ax, y\rangle = \langle x, Ay\rangle$

So A is symmetric.

**Property 6.1.7:** Let A be a symmetric operator and let $\|A\| = \mu = \sup \{\langle Ax, x\rangle; \|x\| = 1\}$

Then atleast one of $-\mu$ is an element of $\sigma(A)$.

**Proof:** Take $x_n$ with $\|x_n\| = 1$ an element of $\sigma(A)$

Then $\exists$ a subsequence $x_{n_k}$ such that $|\langle Ax_{n_k}, x_{n_k}\rangle| \to \mu$.

Clearly $\lambda = \pm\mu$.

Now, $0 \le \left\| Ax_{n_k} \lambda x_{n_k} \right\|^2 = \left\langle Ax_{n_k} - \lambda x_{n_k}, Ax_{n_k} - \lambda x_{n_k} \right\rangle$

$$= \left\| Ax_{n_k} \right\|^2 - \lambda \left\langle Ax_{n_k}, x_{n_k} \right\rangle - \lambda \left\langle x_{n_k}, Ax_{n_k} \right\rangle + \lambda^2 \left\| x_{n_k} \right\|^2$$

$$= \left\| Ax_{n_k} \right\|^2 - \lambda \left\langle Ax_{n_k}, x_{n_k} \right\rangle - \lambda \left\langle Ax_{n_k}, x_{n_k} \right\rangle + \lambda^2 \left\| x_{n_k} \right\|^2$$

$$= \|A\|^2 = -2\lambda \left\langle Ax_{n_k}, x_{n_k} \right\rangle + \lambda^2$$

$$= 2\lambda^2 - 2\lambda \left\langle Ax_{n_k}, x_{n_k} \right\rangle$$

$=2\lambda^2 2\lambda$. $\lambda=0$ as $n\to\infty$

$\therefore \lambda \in \sigma(A)$.

## 6.2 Compact Self Adjoint Operators

**Property 6.2.1:** If A is a compact self adjoint operator, then A has as eigan value $\lambda$ such that $|\lambda| = \|A\|$ and moreover the maximum of $|\langle Ax, x\rangle|$ for $\|x\| = 1$ is achieved on an eigen vector which corresponds to the eigen value $\lambda$

**Proof:** If A = 0, then we can take $\lambda = 0$ and the result holds.

So let $A \neq 0$. Then by 6.17. either $\|A\|$ or $-\|A\|$ is in $\sigma(A)$. So $\exists$

a non-zero $\lambda$ such that $|\lambda|$ is in $\sigma(A)$ and $|\lambda| = ||A||$.

By 5.2.11 $\lambda \in \sigma_p(A)$ since A is compact.

(every non-zero spectral value of a compact operator is an eigen value)

Moreover, if $Ax_0 = \lambda x_0$, where $||x_0|| = 1$ then we have $|\langle Ax_0, x_0\rangle| = |\langle \lambda X0, X0\rangle| = |\lambda|$.

Remark: If T is symmetric, then $\ker T \perp \text{Im} T$.

**Proof**: Given that $\langle Tx, y\rangle = \langle x, Ty\rangle \ \forall\ x, y, \in H$.

Let $x \in \ker T$ and $z \in \text{Im} T$.

$\therefore \exists\ y \in H$ such that $z = Ty$.

Now, $\langle x, z\rangle = \langle x, Ty\rangle = \langle 0, y\rangle = 0$.

$\therefore \ker T \perp \text{Im} T$.

Theorem 6.2.3. (The first Hilbert-Schmidt Theorem)

For every non-zero compact operator. $T: H \to H$ there exist a set of non-zero eigen values $\{\lambda_i\}\ i \geq 1$ in $\mathbb{R}$ such that $|\lambda_1| \geq |\lambda_2| \geq .... \geq |\lambda_i|\ \lambda_{i+1}| \geq ...$ finite or converging to zero and on orthonormal system $\{e_i\} i \geq 1$ of eigen vectors with $Te_i = \lambda e_i$ such that

(i) for all $x \in H$, we have $x + y + \sum_{i \geq 1} \langle x, e_i\rangle e_i$ where $y \in \text{Ker} T$.

(ii) $Tx = \sum i \geq 1 \langle Tx, e_i\rangle e_i = \sum i \geq 1 \langle Tx, e_i\rangle e_i$ and for all $z \in \overline{\text{Im} T}$ we have $z = \sum_{i \geq 1} \langle z, e_i\rangle e$;

i.e. $\{e_i\}_{i \geq 1}$ is an orthonormal basis of $\overline{\text{Im} T}$.

**Proof**: We build the sequence $\{\lambda_i\}$, $\{e_i\}$ by induction

By 6.2.1, $\exists$ an eigen value $\lambda_1$ such that $|\lambda_1| = ||T||$

So clearly, we have an eigen vector $e_1$ such that $||e_1|| = 1$ and $Te_1 = \lambda_1 e_1$.

As induction hypothesis we assume that there exist eigen values $\lambda_1$, $\lambda_2$, ... $\lambda_n$ and eigen vectors $e_1$, $e_2$,....$e_n$ such that $|\lambda_1| \geq |\lambda_2| \geq .... \geq \lambda_n$ | and $||e_i|| = 1$, $Te_i = \lambda_i e_i$ for i= 1,2, ....., n.

Let $E_n$ = span $\{e_1, ...., e_n\}$.

Then $E_n$ is an invariant subspace of T.

Let $x \in E_n$. Then $x = \alpha_1 e_1 + .....+ \alpha_n e_n$.

$Tx = T(\alpha_1 e_1 + ... + (\alpha_1, e_n)$

$= \alpha_1 T(e_1) + ...+ \alpha_n T(e_n)$

$= \alpha_1 \lambda_1 e_1 + ... + \alpha_n \lambda_n e_n \in E_n$

By. 6.1.3, we get that $En^{\perp}$ is an invariant subspace of T.

So T si a symmetric operator on $En^{\perp}$.

Set $T_{n+1} = T | E_n^{\perp}$ ($T_{n+1}$: $En^{\perp}$ is symmetric)

Applying property 6.2.1, there exist an eigen value $\lambda_{n+1}$ and an eigen vector $e_{n+1}$ such that $||e_{n+1}|| = 1$, $Ten_{n+1} = \lambda_{n+1} e_{n+1}$ and $|\lambda_{n+1}| = ||T_{n+1}||$.

Since $E_{n-1} \subseteq E_n$ we have $E_n^{\perp} \subseteq E_{n-1}^{\perp}$.

$\therefore |\lambda n+1| = || T_{n+1}|| = || T| E_n^{\perp}|| \leq ||T/E_{n-1}^{\perp}|| = || T_n || = | \lambda_n|$

So we get $|\lambda_{n+1}| \leq |\lambda_n|$ for any n.

Continuing in this manner, the process either stops when $T_{n+1} = 0$ or we get an infinite sequence $\{\lambda_n\}$ of eigen values of T and corresponding eigen vectors $\{e_n\}$.

For any $x \in H$, set $x_n$ $x_n = \sum_{i=1}^{n} \langle x, e_i \rangle e_i, y_n = x - x_n$

Clearly $y_n \in E_n^{\perp}$.

[Let $y_n \in E_n$. Then $y = \alpha_1 e_1 + ...+ \alpha_n e_n$

$\langle y_n, y \rangle = (x - x_n, y) = \langle x, y \rangle - \langle x_n, y \rangle =$

$$\overline{\alpha_1}\langle x,e_1\rangle+\ldots+\overline{\alpha_n}\langle x,e_n\rangle-\left\langle\langle x,e_1\rangle e_1+\ldots+\langle x,e_n\rangle e_n,\overline{\alpha_1}e_1+\ldots+\overline{\alpha_n}e_n\right\rangle$$

$$= \overline{\alpha_1}\langle x,e_1\rangle+\ldots-\overline{\alpha_n}\langle x,e_n\rangle$$

$$+\ldots\alpha n\langle x,e_n\rangle-$$

$$=0]$$

Case (i) ($T_{n+1} = 0$)

Here, we have $Ty_n = T_{n+1}y_n = 0$. So $y_n \in \ker T$.

Thus we have $x = \sum_{i=1}^{n}\langle x,e_i\rangle e_i + y_n$, where $y_n \in \ker T$.

Case (ii) (The set of eigen values $\{\lambda_n\}$ is infinite:)

By 5.2.4, we have $\lambda_n \to 0$ as $n \to \infty$.

By 2.1.5, $\lim n\to\infty y_n$ exists. Let $y = \lim_{n\to} y_n$.

$\therefore \{y_n\}$ is bounded. Let $\|y_n\| \le M$.

So, $\|Ty_n\| = \|T_{n+1}y_n\| \le \|T_{n+1}\| \|y_n\| \le |\lambda_{n+1}| M \to 0$, as $n \to \infty$.

$\therefore T y_n \to 0$, as $n \to \infty$.

Since $y_n \to y$ and T is bounded, we get $Ty_n \to Ty$.

Thus, we have $Ty = 0$, $\therefore y \in \ker T$.

From $y_n = x - x_n = x - \sum_{i=1}^{n}\langle x,e_i\rangle e_i$, we get taking limit as $n\to\infty$.

$$y = x - \sum_{i=1}^{\infty}\langle x,e_i\rangle e_i$$

i.e; $x = \sum_{i=1}^{\infty}\langle x,e_i\rangle e_i + y$, where $y \in \ker T$

Thus (i) is proved.

Next, we have

$$Tx = \sum_{i=1}^{\infty}\langle x,ei\rangle Te_i + Ty$$

$= \sum_{i=1}^{\infty} \lambda_i \langle x, ei \rangle e_i$

$\therefore$ we get $\text{Im}T \subseteq \overline{span\{e1, e2, ...\}}$

So $\text{Im}T \subseteq span\{e_1, e_2, ...\}$

Since $e_i = \frac{1}{\lambda_i} Te_i \in \text{Im}T, \forall i = 1, 2, ...$ we get that

span $\{e_1, e_2, ...\} \subseteq$ ImT

$\therefore \overline{span\{e_1, e_2, ...\}} \subseteq \overline{\text{Im}T}$

Hence we get that $\text{Im}T = \overline{span\{e_1, e_2, ...\}}$

Thus, by theorem 2.1.10 $\{e_1, e_2, ...\}$ is a basis of $\overline{\text{Im}T}$

$\therefore \forall\ z \in \overline{\text{Im}T}$, we have $z = \sum_{i \geq 1} \langle z, ei \rangle ei.$

The following corollories, T is always assumed to be a self adjoint compact operator in a Hilbert space H and $\lambda_i$, $e_i$ are the eigen values and their corresponding orthonormal eigen vectors which are given by theorem 6.2.3.

Corollary 6.2. 4.

If H is a separable Hilbert space, then exists AN orthonormal basis of eigen vectors $\eta_k$ of the operator T.

**Proof:**

By theorem 6.2.3, $H = \ker T \oplus \overline{\text{Im}T}$

Let kerT$\neq \{0\}$. Consider an orthonormal basis of kerT, say $\{f_i\}$.

Observe that T$f_i$ = 0 = 0. $f_i$. *So* $f_i$ is an eigen vector corresponding to the eigen value 0. Consider the orthonormal basis, we obtained in theorem 6.2.3.

Then $\{e_i\} \cup \{f_i\}$ is an orthonormal basis of eigen vectors.

**Remark**

By theorem 6.2.3., we have $Tx = \sum_{i \geq 1} \lambda_i \langle x, e_i \rangle e_i$

So we have $\langle Tx, x \rangle = \left\langle \sum_{i \geq 1} \lambda_i \langle x, e_i \rangle e_i, x \right\rangle$

$= \sum_{i \geq 1} \lambda_i \langle x, e_i \rangle \langle e_i, x \rangle$

$= \sum_{i \geq 1} \lambda_i \left| \langle x, e_i \rangle \right|^2$

Let us separate the positive and negative eigen values in this expression. We denote $e_i^+$ the eigen vectors corresponding to the positive eigen values $\lambda_i^+$. Similarly the negative eigen values and the corresponding eigen vectors are denoted by $\lambda_\lambda^-$ and $e_i^-$

Then $\sum \lambda_i^+ \left| \langle x, e_i^+ \rangle \right|^2 + \sum \lambda_i^- \left| \langle x, e_i^- \rangle \right|^2 = \langle Tx, x \rangle$

Corollory 6.2.6

Assume that $\lambda_1^+ \geq \lambda_2^+ \geq ...$ and $\lambda_1^- \leq \lambda_2^- \leq ...H$ be an infinite dimensional Hilbert space. Then the following are true,

1. If a positive eigen value exists, then $\max_{\|x\|=1} \langle Tx, x \rangle = \lambda_1^+$

   If a negative eigen value exists, then $\min_{\|x\|=1} \langle Tx, x \rangle = \lambda_1^-$

2. $\langle$ Tx, x$\rangle \geq 0$ for every x$\in$H if and only if there are no negative eigen values.

3. Set $\lambda^+ = \lambda_1^+$ if a positive eigen value exists and $\lambda^+ = 0$ otherwise. Similarly $\lambda^- = \lambda_1^-$ if a negative eigen value exists

and $\lambda^- = 0$ otherwise. Then $\sup_{\|x\|=1} \langle Tx, x \rangle = \lambda^+$ and $\inf_{\|x\|=1} \langle Tx, x \rangle = \lambda^-$ and hence $\lambda^- \|x\|^2 \le \langle Tx, x \rangle \le \lambda^+ \|x\|^2$

**Proof**

(i) assume that a positive eigen value exists.

we have $\langle Tx, x \rangle = \sum \lambda_i^+ \left| \langle x, e_i^+ \rangle \right|^2 + \sum \lambda_i^1 \left| \langle x, e_i^- \rangle \right|^2$

$\le \sum \lambda_i^+ \left| \langle x, e_i^+ \rangle \right|^2$

$\le \lambda_i^+ \sum \left| \langle x, e_i^+ \rangle \right|^2$

$\le \lambda_i^+ \|x\|^2$ (by Bessel's inequality)

that is $\langle Tx, x \rangle \le \lambda_1^+$ for every x$\in$H with ||x||-1

So $\sup_{\|x\| = 1} \langle Tx, x \rangle \le \lambda_1^+$

Next for $x = e_1^+$ we have $\langle Te_1^+, e_1^+ \rangle = \langle \lambda_1^+, e_1^+, e_1^+ \rangle = \lambda_1^+$

thus we get $\max_{\|x\|=1} \langle Tx, x \rangle = \lambda_1^+$

Similarly assume that a negative eigen value exists.

We have $\langle Tx, x \rangle = \sum \lambda_i^+ \left| \langle x, e_i^+ \rangle \right|^2 + \sum \lambda_i^- \left| \langle x, e_i^- \rangle \right|^2$

$\ge \sum \lambda_i^- \left| \langle x, e_i^- \rangle \right|^2$

$\ge \lambda_i^- \sum \left| \langle x, e_i^- \rangle \right|^2$

$\ge \lambda_i^- \|x\|^2$

that is $\langle Tx, x\rangle \geq \lambda_1^-$ for every x∈H ||x|| = 1

So $\inf_{\|x\|=1}\langle Tx,x\rangle \geq \lambda_1^-$

Next for x = $e_1^-$ we have $\langle Te_1^-, e_1^-\rangle = \langle \lambda_1^- e_1^-, e_1^-\rangle = \lambda_1^-$

thus we get $\min_{\|x\|=1}\langle Tx,x\rangle = \lambda_1^-$

(ii) assume that $\geq 0$ for every x∈H

This implies that $\min_{\|x\|=1}\langle Tx,x\rangle = \geq 0$

If possible let there be a negative eigen value.

Then by (i) $\min_{\|x\|=1}\langle Tx,x\rangle = \lambda_1^- < 0$, which is a contradiction.

Then there does not exist a negative eigen value.

Conversely, that there be no negative eigen value.

$\langle Tx,x\rangle = \Sigma\lambda_i^+ |\langle x, e_i^+\rangle|^2 \geq 0$ for every x∈H

(iii) If a positive eigen value exists, then $\sup_{\|x\|=1}\langle Tx,x\rangle = \lambda_1^+ = \lambda^+$

If a positive eigen value does not exists, then $\langle Tx,x\rangle \leq 0$

We know that $\lambda_i \to 0$ as $i \to \alpha$ (or become 0 after some i)

We can take $x_n = e_n^-$ and obtain $\langle Tx_n,x_n\rangle = \langle Te_n^-, e_n^-\rangle = \overline{\lambda_n} = \lambda^-$

Thus we get that $\sup_{\|x\|=1}\langle Tx, x\rangle = 0 = \lambda^+$

Similarly if a negative eigen value exists, then $\inf_{\|x\|=1}\langle Tx,x\rangle = \lambda_1^- = \lambda^-$

If a negative eigen value does not exists, then $\langle Tx,x\rangle \geq 0$

We know that $\lambda_i \to 0$ as $i \to \alpha$ (ir become 0 after some i)

We can taken $x_n = e_n^-$ and obtain $\langle Tx_n,x_n\rangle = \langle Te_n^+, e_n^+\rangle =$

$\lambda_n^+ \to 0$

Thus we get that $\inf_{\|x\|=1} \langle Tx,x\rangle = 0 = \lambda^-$

For every non zero $x \in H$, we have,

$\lambda_- \le \left\langle T(\frac{x}{\|x\|}), \frac{x}{\|x\|} \right\rangle \le \lambda^+$ This gives $\lambda^- \|x\|^2 \le \langle Tx,x\rangle \le \lambda^+ \|x\|^2$

For x = 0 the above inequality is obviously true. Hence

$\lambda^-\|x\|^2 \le \langle Tx,x\rangle \le \lambda^+\|x\|^2$ for every $x \in H$

**Corollory 6.2.7 (Minimax principle)**

Let $\lambda_{n+1}^+ > 0$ (meaning that there are atleast n + 1 positive eigen values).

Then $\lambda_{n+1}^+ = \min_{\{x1,\ldots xn\}} \phi(x_1,\ldots,x_n)$ where

$\varnothing(x_1,\ldots x_n) = \sup_{\|x\|=1} \{\langle Tx,x\rangle ;\ x \in (span\{x_i\}_1^n)^\perp \}$

**Proof**

Let $x_1, \ldots, x_n \in H$

**Claim**: There exists a y such that $y \perp x_1, \ldots, x_n$

$\|Y\| = 1$ and $y \in$ span $\{e_1^+, e_2^+, \ldots e_{n+1}^+$ such that

$\langle y,x_1\rangle = 0, \langle y,x_2\rangle = 0, \langle y, x_n\rangle = 0$

i.e., we try to find $b_1, b_2 \ldots, b_{n+1}$ such that

$b_1\langle e_1^+, x_1\rangle + b_2\langle e_2^+, x_1\rangle + \ldots + b_{n+1}\langle e_{n+1}^+, x_1\rangle = 0$

$b_1\langle e_1^+, x_2\rangle + b_2\langle e_2^+, x_2\rangle + \ldots + b_{n+1}\langle e_{n+1}^+, x_2\rangle = 0$

.

.

.

$b_1\langle e_1^+, x_n\rangle + b_2\langle e_2^+, x_n\rangle + .. + b_{n+1}\langle e_{n+1}^+, x_n\rangle = 0$

This is a homogeneous system of n equations with n+1 unknowns.

$b_1, b_2, \ldots b_{n+1}$. This system has a non trivial solution $b_1, b_2, \ldots b_{n+1}$.

Let $b = |b_1|^2 + |b_2|^2 + \ldots + |b_{n+1}|^2)^{1/2}$

Then $a_1 = \frac{b_1}{b}, a_2 = \frac{b_2}{b} \ldots, a_{n+1} = \frac{b_{n+1}}{b}$ is a non trival solution of the system and moreover $|a_1|^2 + |a_2|^2 + \ldots + |a_{n+1}|^2 = 1$

So we take $y = a_1e_1^+ + a_2e_2^+ + \ldots + a_{n+1}e_{n+1}^+$

$\|y\|^2 = \langle y, y\rangle =$

$\left\langle a_1e_1^+ + a_2e_2^+ + \ldots + a_{n+1}e_{n+1}^+, a_1e_1^+ = a_2e_2^+ + \ldots. + a_{n+1}e_{n+1}^+ \right\rangle$

$= -|a_1|^2 + |a_2|^2 + \ldots + |a_{n+1}|^2 = 1$

Hence our claim is proved

Then $\varnothing(x_1, \ldots x_n) \geq (Ty, y)$

$= \left\langle T(a_1e_1^+ + a_2e_2^+ + \ldots + a_{n+1}e_{n+1}^+), a_1e_1^+ + a_2e_2^+ + \ldots. + a_{n+1}e_{n+1}^+ \right\rangle$

$= \left\langle a_1\lambda_1^+e_1^+ + \ldots + a_1\lambda_1^+e_1^+, a_1e_1^+ + a_2e_2^+ + \ldots. + a_{n+1}e_{n+1}^+ \right\rangle$

$\geq |a_1|^2 \lambda_{n+1}^+ + |a_2|^2 \lambda_{n+1}^+ + |a_{n+1}|^2 \lambda_{n+1}^+$

$= \lambda_{n+1}^+$

This gives $\min_{\{x1, \ldots xn\}} \phi(x_1, \ldots, x_n) \geq \lambda_{n+1}^+$

Now taking $x_1 = e_1^+, x_2 = e_2^+, \ldots, x_n = e_n^+$ we have,

$\varnothing(x_1, ..... x_n) = \varnothing(e_1^+, e_2^+, ...., e_n^+)$

$= \sup_{\|x\|=1}\{\langle Tx,x\rangle; x\in(\text{span}\{e_1, e_2, ...., e_n\})^{\perp}\}$

$= \sup_{\|x\|=1}\{\langle Tx,x\rangle; x\in E^{\perp}_n\}$

$= \sup_{\|x\|=1}\{\langle T_{n+1}x,x\rangle; x\in E^{\perp}_n\}$

$= \lambda^+_{n+1}$ (by corollary 6.2.6 applied to the operator $T_{n+1}$)

Hence $\min_{\{x_1,....,x_n\}} \phi(x_1,...,x_n) = \lambda^+_{n+1}$

**Theorem 6.2.8 (The second Hilbert-Schmidt theorem; a theorem on symmetric kernels)**

Let $H = L_1[a,b]$. Consider the operator

$$(Tx)(s) = \int_a^b k(s,t)x(t)dx \text{ where } k(s,t) \in L_2(I^2), I = [a,b]$$

It is known that compact operator.

Let $T = T^*$ (meaning that $k(s,t) = \overline{k(s,t)}$)

Let $\{e_i(t)\}$ be all orthonormal eigen vectors of T (according to the first Hilbert-Schmidt theorem) and $Te_i = \lambda_i e_i$

Then $k(s,t) = \sum_{i\geq 1} \lambda_i e_i(s)\overline{e_i(t)}$

(The convergence of the series and the equality are understood in the sense of $L_2(I^2)$).

as a consequence, $\sum \lambda_i^2 = \int_a^b\int_a^b |k(s,t)|^2 \, dsdt < \infty$

**Proof**

Let $\eta_i(s,t) = e_i(s)\overline{e_i(t)}$

Then $\{\eta_i\}$ is an orthonormal system in $L_2(I^2)$

For $i \neq j$,

$\langle \eta_i,\eta_j\rangle = \int_a^b\int_a^b | n_i\overline{n_j} dsdt$

$= \int_a^b \int_a^b |e_i(s)|^2 |e_i(t)|^2 \, dsdt$

$= \int_a^b \int_a^b |e_i(s)|^2 | ds f_a^b e_i(t)|^2 \, dsdt$

= 1.1 = 1

Hence the function ∅ (s,t) = $\sum i \geq 1 \langle k, \eta_i \rangle \eta_i$ exists as an element of $L_2(I^2)$ i.e. the series converges

Now $\langle k, n_i \rangle = \int_a^b \int_a^b k(s,t) e_i(t) dt) e_i(s) e_i(t) dsdt$

$= \int_a^b \int_a^b k(s,t) e_i(t) dt) \overline{e_i(s)} ds$

$= \int_a^b \int_a^b T \overline{e_i e_i(s)} ds$

$= \langle Te_i, e_i \rangle L_{2(x)}$

$= \lambda_i \langle e_i e_i \rangle$

$= \lambda_i$

Then $(s,t) = \sum_{i \geq 1} \lambda_i n_i = \sum_{i \geq 1} \lambda_i e_i(s) \overline{e_i(t)}$

Take $x(t) \in L_2(x)$. By the first Hilbert-Schmidt theorem, we know that

$Tz = \sum (Tz, x) = \int_a^b \int_a^b k(s,t) z(t) \overline{x(s)} dtds$

and $\langle k, x(s) \overline{z(t)} \rangle L_2(I^2)$

$\langle Tz, x \rangle = \langle \Sigma [Tz, e_i] e_i, x \rangle$

$= \Sigma \langle Tz, e_i \rangle, \langle e_i, x \rangle$

$= \Sigma \langle z, Te_i \rangle, \langle e_i, x \rangle$

$= \Sigma \langle z, \lambda_i e_i \rangle \langle e_i, x \rangle$

$= \Sigma\lambda_i\langle z, e_i\rangle \langle e_i, x\rangle$

We note that $[n_i(s,t), x(s)\overline{z(t)}] = \int_a^b \int_a^b e_i(s)\overline{e_i(t)}z(t)\overline{x(s)}dsdt$

$\int_a^b z(t)\overline{e_i(t)}dt \int_a^b e_i(s)\overline{x(s)}ds$

$= \langle z, e_i\rangle\langle e_i, x\rangle$

Thus we get $\langle Tz, x\rangle = \sum \lambda_i \langle \eta_i, x\bar{z}\rangle$

$= \sum \langle \lambda_i n_i, x\bar{z}\rangle$

$= \langle \phi, x\bar{z}\rangle$

Consequently for every function x(s) and x(t) in $L_2(I)$ we have

$\langle \phi, x\bar{z}\rangle = \langle k, x\bar{z}\rangle$

i.e. $\langle \phi - k, x\bar{z}\rangle = 0$

Note that $\{x(s)\overline{z(t)}; x \in L_2(I), z \in L_2(I)\}$ is complete in $L_2(I^2)$

Then $\phi - k, x = 0$

So $\phi = k$

Hence k (s, t) = $\sum \lambda_i n_i = \sum \lambda_i e_i(s)\overline{e_i(t)}$

Consequently

$\| k(s,t)\|^2 = \int_a^b \int_a^b |k(s,t)|^2 \, d$sdt and $\|\sum \lambda_i \eta_i\|^2 = \sum |\lambda_i|^2$ since *n* is an o.n, system.

Thus we get $\sum |\lambda_i|^2 = \int_a^b \int_a^b |k(s,t)|^2 \, dsdt < \alpha$

**Remark**

The integral operators defined by the kernel functions k(s, t) $\varepsilon L_2 (I)^2$ with k(s, t) = k (t, s) lead to small subclass of compact symmetric operators. Their set of eigen values must tend to 0 or quickly that $\sum |\lambda_i|^2 < \infty$. For example, there is no such operator with eigen values $\lambda_n = I/\sqrt{n}$ (For

$\lambda_n = 1/\sqrt{n}, \sum \lambda_n^2 = \sum 1/\sqrt{n}$)

**Theorem 6.2.11 (Mercer theorem)**

Let k(s, t) be continuous and assume that the all of the eigen values $\lambda_i$ of the corresponding integral operator T are non-negative (i.e $\langle Tx, x\rangle \geq 0$ for all x). Then $k(s,t) = \sum_{i\geq 1} \lambda_i e_i(s) e_i(t)$ and this series converges absolutely and uniformly.

**Corollory 6.2.12**

Under the above conditions, $\sum \lambda_i = \int_a^b k(t,t)dt$ and k(t, t) $\geq 0$

**Proof**

By Mercer's theorem, $k(t,t) = \sum \lambda_i e_i(s)\overline{e_i(t)} = \sum \lambda_i |e_i(t)|^2$

i.e. $k(t,t) \geq 0$ (since $\lambda_1$'s are non - negative)

Also, $\int_a^b k(t,t)dt = \int_a^b \sum \lambda_i |e_i(t)|^2 \, dt$

$\sum \lambda_i \int_a^b |e_i(t)|^2 \, dt$

$\sum \lambda_i .1$

$\sum \lambda_i \left( \text{since} \int_a^b |e_i(t)|^2 \, dt = 1 \right)$

## 6.3 Order in the space of self-adjoint operators

Definition 6.3.1

An operator A is called non-negative (we write $A \geq 0$) if and only $(Ax, x) \geq 0$ for all $x \in H$.

If A is no-negative, then clearly A is symmetric.

Moreover, $A \leq B$ means that

(1) Both A and B are symmetric

(2) $B - A \geq 0$

**Properties of the ordering**

(1) If $-1 \leq A \leq 1$, then $\|A\| \leq 1$

Proof: $-1 \leq A \leq 1 \Rightarrow (-1x,x) \leq (Ax, x) \leq (Ix, x)\ \forall x \in H$

$\Rightarrow -\|x\|^2 \leq (Ax, x) \leq \|x\|^2 \forall x \in H$

$$\Rightarrow -1 \leq \frac{\langle Ax, x\rangle}{\|x\|^2} \leq 1, \forall x \neq 0$$

i.e. $$\frac{\langle Ax, x\rangle}{\|x\|^2} \leq 1, \forall x \neq 0$$

$$\therefore \sup_{x \neq 0} \frac{\langle Ax, x\rangle}{\|x\|^2} \leq 1$$

i.e, $\|A\| \leq 1$

Note : If A is a self adjoint operator, then $-1\|A\| \leq A \leq 1\|A\|$

Proof: We have $\sup \frac{|\langle x, x\rangle|}{\|x\|^2} = \|A\| \Rightarrow \frac{|\langle x, x\rangle|}{\|x\|^2} \leq \|A\|, \forall x \neq 0$

$$\Rightarrow -\|A\| \le \frac{\langle Ax, x\rangle}{\|x\|^2} \le \|A\|, \forall x \neq 0$$

$$\Rightarrow -\|A\|\langle Ix, x\rangle \le \langle Ax, x\rangle \le \|A\|\langle Ix, x\rangle, \forall x \in H$$

i,e,

$= \|A\|\langle Ix, x\rangle \le \langle Ax, x\rangle \le \|A\|\langle Ix, x\rangle, \forall x \in H \therefore weget -\|A\|I \le A \le \|A\|I$

(2) If $A \ge 0$ and $A \le 0$ then $A = 0$

Proof : since $A \ge 0$, we have $(Ax, x) \ge 0$, $\forall x \in H$; and since $A \le 0$, we have $(Ax, x) \le 0$, $\forall x \in H$.

$\therefore \langle Ax, x\rangle = 0, \forall X \in H$

we have $\|A\| = \sup_{x \neq 0} \frac{|\langle Ax, x\rangle|}{\|x\|^2} = 0$

$\therefore$we get $A = 0$

(3) Generalized cauchy-schwartz inequality:

Let $A \ge 0$ then $|\langle Ax, y\rangle| \le \sqrt{\langle Ax, x\rangle}\sqrt{\langle Ay, y\rangle}$

[Let $\langle\langle x, y\rangle\rangle$ be a quasi-inner product. i.e, it stisfies all the conditions of the inner product except that $\langle\langle x, x\rangle\rangle$ may be zero for a non-zero element.

Then caushy-schwartz inequality is still true, i.e., $|\langle\langle x, y\rangle\rangle| \le \langle\langle x, x\rangle\rangle^{\frac{1}{2}}\langle\langle y, y\rangle\rangle^{\frac{1}{2}}$]

Proof:- If $\langle\langle x, y\rangle\rangle = 0$ , then the inequality is true. Let $|\langle\langle x, y\rangle\rangle| \neq 0$

Set $\theta = \dfrac{\langle\langle x, y\rangle\rangle}{\left|\langle\langle x, y\rangle\rangle\right|}$ Let $\lambda$ be a real number. We have

$$0 \le \left\langle\left\langle \bar{\theta}x + \lambda y, \theta x + \lambda y\right\rangle\right\rangle = \bar{\theta}\theta\langle\langle x, x\rangle\rangle + \bar{\theta}\lambda\langle\langle x, y\rangle\rangle + \lambda\theta\langle\langle y, x\rangle\rangle + \lambda^2\langle\langle y, y\rangle\rangle$$

$$= \langle\langle x, x\rangle\rangle + \lambda\left|\langle\langle x, y\rangle\rangle\right| + \lambda\left|\langle\langle x, y\rangle\rangle\right| + \lambda^2\langle\langle y, y\rangle\rangle$$

$$\langle\langle x, x\rangle\rangle + 2\lambda\left|\langle\langle x, y\rangle\rangle\right| + \lambda^2\langle\langle y, y\rangle\rangle \to (A)$$

Put $\lambda = \dfrac{\langle\langle x, x\rangle\rangle}{\left|\langle\langle x, y\rangle\rangle\right|}$, we get that

$$0 \le \langle\langle x, x\rangle\rangle - 2\langle\langle x, x\rangle\rangle + \frac{\langle\langle x, x\rangle\rangle^2\langle\langle y, y\rangle\rangle}{\left|\langle\langle x, y\rangle\rangle\right|^2}\ \mathbb{N}$$

i.e, $\langle\langle x, x\rangle\rangle\left|\langle\langle x, y\rangle\rangle\right|^2 \le \langle\langle x, x\rangle\rangle^2\langle\langle y, y\rangle\rangle$

i.e, $\left|\langle\langle x, y\rangle\rangle\right|^2 \le \langle\langle x, x\rangle\rangle\langle\langle y, y\rangle\rangle$, if $\langle\langle x, x\rangle\rangle \ne 0$

If $\langle\langle x, x\rangle\rangle = 0$ then we have by (A)

$0 \le 2\lambda\left|\langle\langle x, y\rangle\rangle\right| + \lambda^2\langle\langle y, y\rangle\rangle$, for every $\lambda \in \mathbb{R}$

Take $\lambda = \dfrac{-1}{n}, n \in \mathbb{N}$ we get $0 \le \dfrac{-2}{n}\left|\langle\langle x, y\rangle\rangle\right| + \dfrac{1}{n^2}\langle\langle y, y\rangle\rangle$

i.e, $2\left|\langle\langle x, y\rangle\rangle\right| \le \dfrac{1}{n}\langle\langle y, y\rangle\rangle$, for every $n \in \mathbb{N}$.

This gives $2\left|\langle\langle x, y\rangle\rangle\right| = 0 i.e, \left|\langle\langle x, y\rangle\rangle\right| = 0$

$\therefore \langle\langle x, y\rangle\rangle = 0$

So both sides of cauchy-schwartz inequality are zerol.]

$\langle\langle x, y\rangle\rangle = \langle Ax, y\rangle$. Then $\langle\langle ,\rangle\rangle$ is a quasi inner product.

[$\langle\langle x, x\rangle\rangle = \langle Ax, x\rangle \geq 0$ since A $\geq 0$

If x = 0, then $\langle\langle x, x\rangle\rangle = \langle A0,0\rangle = \langle 0,0\rangle = 0$

$\langle\langle x, y\rangle\rangle = \langle Ax, y\rangle = \langle x, Ay\rangle$ since A is symmetric.

$$\langle Ay, x\rangle = \langle\langle y, x\rangle\rangle$$

$$\langle\langle ax_1 + bx_{2,} y\rangle\rangle = \langle A(ax_1 + bx_2, y)\rangle = \langle aA(x_1 + bA(x_2), y)\rangle$$

$$= a\langle A(x_1), y\rangle + b\langle A(x_2), y\rangle$$

$$= a\langle\langle x_1, y\rangle\rangle + b\langle\langle x_2, y\rangle\rangle]$$

Then by cauchy schwartx inequality, for quasi inner product, we get $|\langle Ax, y\rangle| \leq \sqrt{\langle Ax, x\rangle}.\sqrt{\langle Ay, y\rangle}$

(4) If C is symmetric and A ≤ B, then A + C ≤ B + C

[Since A ≤ B, we have $\langle (B-A)x, x\rangle \geq 0$

$\langle Bx - Ax, x\rangle \geq 0 \Rightarrow \langle Bx, x\rangle \geq \langle Ax, x\rangle$

$\langle Bx, x\rangle + \langle Cx, x\rangle \geq \langle Ax, x\rangle + \langle Cx, x\rangle$

i.e, $\langle (B+C)x, x\rangle \geq \langle (A+C)x, x\rangle$

So $B + C \geq A + C$

(5) If A is symmetric, then $A^{2n} \geq 0$.

[$\langle A^{2n}x, x\rangle = \langle A^n x, A^n x\rangle = \|A^n x\|^2 \geq 0$]

If $A \geq 0$, then $A^{2n+1} \geq 0$

$\left[\left\langle A^{2n+1}x, x\right\rangle = \left\langle AA^n x, A^n x\right\rangle \geq 0\right]$

**Note:**

If $A \geq 0$, then for any polynomial $p(\lambda)$ with non negative coefficients we have $p(A \geq 0$.

**Theorem 6.3.2.** (On convergence of monotone and bounded sequence of operators)

Let $A_0 \leq A_1 \leq \ldots\ldots\ldots \leq A_n \leq \ldots\ldots \leq A$.

Then there exists a strong limit of $(A_n)$.

(i.e, $\exists$ a bounded operator B such that $A_n x \rightarrow Bx$ for all $x \in H$)

Proof: For every symmetric operator A, there is a number $C > 0$ such that $A \leq C.1$]

Since $A - A_0$ is symmetric $\exists_{c1} > 0$ such that $A - A_0 \leq c_1 I$

without loss of generality, assume that $0 \leq A_n \leq I$

[Instead of $A_n$, we consider $\frac{A_n - A_0}{c_1}$

we have $A_n - A_0 \leq A - A_0 \forall n = 1,2,\ldots\ldots$

$\leq c_1 I$

$\therefore \frac{A_n - A_0}{c_1} \leq I$

Since $A_n \geq A_0, \forall n = 1,2,\ldots\ldots$ we have $\frac{A_n - A_0}{c_1} \geq 0$

So we have $0 \leq \frac{A_n - A_0}{c_1} \leq I$]

For $n > m$, define $A_{mn} = A_n - A_m \geq 0$

we have $\|A_{mn}\| \leq 1 \text{since} 0 \leq A_{mn} \leq I$

For any x and y = $A_{mn}^{x,}$ we have

$$\|A_n x - A_m x\|^2 = \|(A_n - A_m)x\|^2 = \|A_{mn}x\|^2$$

$$= \langle A_{mn}x, A_{mn}x \rangle$$

$$= \langle A_{mn}x, y \rangle$$

$$\leq \sqrt{\langle A_{mn}x, x \rangle} \sqrt{\langle A_{mn}y, y \rangle}$$

$$= \sqrt{\langle A_{mn}x, x \rangle} \sqrt{A_{mn}^2 x, A_{mn}x}$$

$$\leq \sqrt{\langle A_{mn}x, x \rangle} \sqrt{\|A_{mn}^2 x\| \|A_{mn}x\|}$$

$$\leq \sqrt{\langle A_{mn}x, x \rangle} \sqrt{\|x\| \|x\|}$$

$$= \sqrt{\langle A_{mn}x, x \rangle} \|x\|$$

Thus $\|A_n x - A_m x\|^2 \leq \sqrt{\langle A_n x, x \rangle - \langle A_m x, x \rangle}.\|x\|$

$$\to 0 as \, n > m \to \infty, \text{ for all } x \in H.$$

[($A_n$x, x) is a monotone increasing sequence, which is bounded above by (Ax, x) and therefore it converges and hence cauchy. So ($A_n$x, x) - ($A_m$x, x) → 0]

So ($A_n$x) is a cauchy sequence in H and hence converges.

we fedine B : H → H as $Bx = lim\ A_n x$

clearly B is linear.

moreover, $0 \leq \langle A_n x, x \rangle \sin ce\ A_n \geq 0$

$$\leq \langle Ix, x \rangle \text{since} A_n \leq I$$

$$= \langle x, x \rangle$$

Talking limits, we get

$$0 \le \langle Bx, x \rangle \le \|x\|^2$$

$\therefore$B is bounded operator.

**Lemma 6.3.5:** Let A $\geq$ 0. Then there exists an operator X such that $X^2$ = A and X $\geq$ 0. (we write $\sqrt{A}$ for X)

Moreover, for all B such that AB = BA, it is also true that $\sqrt{AB} = B\sqrt{A}$

**Proof:** w.l.g assume that 0 ≤ A ≤ I.

(we need to find X $\geq$ 0 such that $X^2$ = A) (Otherwise divide A by ||A||

if $T = \frac{A}{\|A\|}$, then we have (Ax, x) $\le \|A\| \|x\|^2$

i.e., $\left\langle \frac{A}{\|A\|} x, x \right\rangle \le \|x\|^2$

i.e, $\langle Tx, x \rangle \le \langle Ix, x \rangle$

So T $\leq$ I)

Let B = I - A and Y = I - X.

Then A = I - B, x = I -Y, 0 $\leq$ B $\leq$ I and the equation to be solved is

$$(I - Y)^2 = 1 - B \text{ i.e, } I - 2Y + Y^2 = I - B, i.e, Y = \frac{B + Y^2}{2}$$

Define $Y_0 = 0, T_{n+1} = \frac{1}{2}\left(B + Y_n^2\right)$, forn = 0, 1, 2, .....

It is clear that $Y_n \geq 0$ and $Y_n$ is a polynomial in B withy non-negative coefficients for all $n \in \mathbb{N}$

claim: $Y_n \leq I, \forall n \in \mathbb{N}$

Proof : we use induction. Clearly $Y_0 \leq I$ and $Y_1 \leq I$.

As induction hypothesis, we assume $Y_{n-1} \leq I$

we have $\|Y_{n-1}\| \leq 1$ since $0 \leq Y_{n-1} \leq I$

$$\therefore \left(Y_{n-1}^2, x\right) = \left\langle Y_{n-1}x, Y_{n-1}x \right\rangle = \left\|Y_{n-1}x\right\|^2 \leq \left\|Y_{n-1}\right\|^2 \left\|x\right\|^2$$

$$\leq \left\|x\right\|^2$$

$$= \left\langle Ix, x \right\rangle$$

So we have $Y_{n-1}^2 \leq I$

$$\therefore Y_n = \frac{1}{2}\left(B + Y_{n-1}^2\right) \leq \frac{1}{2}(1+1) = 1$$

$\therefore$by induction the claim holds.

claim: $Y_{n+1} - Y_n \geq 0$. Moreover, they are polynomials in B with non-negative coefficients.

proof: We again use induction.

clearly $Y_2 - Y_1 \geq 0$ and is a polynomial in B with non negative coefficients.

$$\left[ Y_2 - Y_1 = \frac{1}{8} B^2 \right]$$

As induction hypothesis, assume $Y_n - Y_{n-1}$ is a polynomial in B with non negative coefficients.

Now $Y_{n+1} - Y_n = \frac{1}{2}\left(Y_n^2 - Y_{n-1}^2\right) = \frac{1}{2}\left(Y_n^2 + Y_{n-1}^2\right)\left(Y_n - Y_{n-1}\right)$ [We use

here that $Y_nY_{n-1} = Y_{n-1}Y_n$ because they are polynomials of the same operator B.]

Also $Y_n + Y_{n-1}$ is a polyomial with non negative coefficients.

$\therefore Y_{n+1} - Y_n$ is a polynomial of B with non-negative coeffients and as a result $Y_{n+1} - Y_n \geq 0$ for every n = 0, 1, 2, ....

Thus our claim is proved.

Hence by theorem 6.3.2, $\exists$ an operator $Y\infty$ such that $Y_n \to Y_\infty$ strongly.

Since $Y_{n+1} = \frac{1}{2}\left(B + Y_n^2\right)$, we get that $Y_\infty = \frac{1}{2}\left(B + Y_\infty^2\right)$.

$\therefore$We can take X=1 - Y$\infty$ as $\sqrt{A}$

Since $0 \leq Y_\infty \leq 1$, we get that $Y_\infty = \frac{1}{2}\left(B + Y_\infty^2\right)$

$\therefore$we can takes X = 1 - $Y_\infty$ as $\sqrt{A}$

Since $0 \leq Y_\infty \leq 1$, we get that $X \geq 0$

X is a strong limit of polynomials of B (an also of A)

$\therefore$for all c such that Ac = cA, we have

p(A)c = cp(A) which implies Xc = cX

**Corollary 6.3.6:** If $A \geq 0$ and (Ax, x) = 0, then Ax=0

proof: $(Ax, x) = 0 \Rightarrow \left\langle \sqrt{A}\sqrt{Ax}, x \right\rangle = 0 \Rightarrow \left\langle \sqrt{Ax}, \sqrt{Ax} \right\rangle = 0$

$$\Rightarrow \sqrt{Ax} = 0 \Rightarrow \sqrt{A}\left(\sqrt{Ax}\right) = 0 \Rightarrow Ax = 0$$

Remark: The positive square root of $A \geq 0$, given by lemma 6.3.5, is unique.

i.e., if $X_1^2 = A$ then $X_1 = X$

proof: First we note that $XX_1 = X_1X$

[we have $X_1A = X_1^3 = AX_1 \therefore by$ 6.3.5 $XX_1 = X_1X$ ]

If $y = (X - X_1)x$, then $0 = \langle (X + X_1)y, y \rangle = \langle Xy, y \rangle + \langle X_1y, y \rangle$

since both of these terms are no negative, we get $Xy = 0$ and $X_1y = 0$

Hence $X^2 = XX_1 = 4$

[Let $y = (X - X_1)x$. Then

$\langle Xy, y \rangle + \langle X_1y, y \rangle = \langle (X + X_1)y, y \rangle =$

$\langle (X + X_1)(X - X_1)x, y \rangle = \langle (X^2 - X_1^2)x, y \rangle = \langle 0, y \rangle = 0$

$\Rightarrow$(Xy, y) = 0 and ($X_1$y, y) = 0, since both terms are non negative.

Therefore by corollary 6.3.6, we get $Xy = 0$ and $X_1y = 0$.

Now, $Xy = 0 \Rightarrow X(X - X_1)x = 0 \Rightarrow (X^2 - XX_1)x = 0 \Rightarrow X^2x = XX_1x \Rightarrow X^2 = XX_1$.]

Now, $\forall x$, we have $\|(X - X_1)x\|^2 =$

$\langle (X - X_1)x, (X - X_1)x \rangle = \langle (X - X_1)^2 x, x \rangle =$

$\langle (X^2 - 2XX_1 + X_1^2)x, x \rangle = \langle (A - 2A + A)x, x \rangle = \langle 0x, x \rangle = 0$

This gives $(X - X_1)x = 0$, $\forall x$

$\therefore X - X_1 = 0$ and hence $X = X_1$.

**Lemma 6.3.4:** If $A \geq 0, B \geq 0$ and AB = BA, then AB $\geq$ 0.

proof: By lemma 6.3.5, there exists positive square root $\sqrt{A}$ for A $\geq$ 0.

We have (ABx, x) =

$$\left\langle \sqrt{A}\sqrt{AB}x, x \right\rangle = \left\langle \sqrt{A}Bx, \sqrt{A}x \right\rangle = \left\langle B\sqrt{A}x, \sqrt{A}x \right\rangle \geq 0$$

Hence we get AB $\geq$ 0.

**Proposition 6.3.3**: Let A be such that mI $\leq$ A $\leq$ MI for some m, M $\in \mathbb{R}$ and let P be a polynomial satisfying P(z) $\geq$ 0 for all z $\in$ [m, M]. Then P (A) $\geq$ 0.

**Proof:**

We may assume m < M.

(Let m=M. Then A=ml. Let P(z) $= a_0 + a_1 z + a_2 z^2 + ..... + a_n z^n$

$P(m) \geq 0 \Rightarrow a_0 + a_1 m + .....+ a_n m^n \geq 0$

Now, $P(A) = P(mI) = a_0 I + a_1 (mI) + a_2 (mI)^2 + .... + a_n (mI)^n$

$$= a_0 + a_1 mI + a_2 m^2 I + ....+ a_n m^n I$$

$$= \left(a_0 + a_1 m + a_2 m^2 + ...+ a_n m^n\right) I \geq 0.)$$

Since P(z) $\geq$ 0 for z $\in$[m, M], we get that P(z) has real coefficients and it may be written in the form.

$$P(z) = c. \prod_{\alpha_i \leq m} (z - \alpha_i) \prod_{\beta_i \geq M} \left(\beta_i - z\right). \prod \left\langle \left(z - \gamma_i\right)^2 + \delta_i^2 \right\rangle,$$

where $\gamma_i, \delta_i \in R$.

$\left(LetP(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n\right) = \left(x_0 + x_1 z + ... + x_n z^n\right) +$

$i\left(y_0 + y_1 z + .... + y_n z^n\right).$

$P(z) \geq 0 \forall z \in [m, M] \Rightarrow y_0 + y_1 z + y_2 z^2 + ... + y_n z^n = 0$

$\Rightarrow y_0 + y_1 z + y_2 z^2 + ... + y_n z^n$ must be the zero polynomial. Since a non zero polynomial cannot have infinitely many zeros).

Indeed, $\alpha_i$ are the real roots to the left of m, $\beta_i$ are the real roots to the right of M, and the quadratic factors take care of all the real roots on (m, M) and all the complex roots.

Obviously $A - \alpha_i I \geq 0, \beta_i I - A \geq 0$ and $\left(A - \gamma_i I\right)^2 + \delta_i^2 I \geq 0.$

Since all these opperators are pairwise commutative their product is also non-negative (by lemma 6.3.4).

Corollary 6.3.7: If $mI \leq A \leq MI$ and $P_1(t)$, $P_2(t)$ are real polynomials with

$P_1(t) \leq P_2(t)$ for all $t \in [m, M]$, then $P_1(A) \leq P_2(A)$

Proof: (Apply the above result for $P_2 - P_1(t)$.)

## 6.4.Projection Operatros

Let E be a linear space. A linear operator $P.E \rightarrow E$ is called a projection if and only if $P^2 = P$

Define $E_1 = ImP$, $E_2 = ker\ P$.

**Properties of projection operators:**

**Proposition 6.4.1**: $P/_{E1} = Id_{E1,}$ *ie;* $Px=x, \forall x \in E_1$

Proof: Let $x \in E_1$. Then $\exists\ y \in E$ such that $Py=x$.

Now, $Px=P(Py)=P^2y = Py = Py = x$.

Proposition 6.4.2: Theoperator $Q=I-P$ isaprojection (ie; $Q^2 = Q$) andIm $P=\ker Q$, $\ker P=1m\ Q$.

Proof : Clearly $Q^2 = (I-P)^2 = I^2 - 2P + P^2 = I - 2P + P = Q$. So Q is a projection.

Let $x \in$ ImP, then we have $Px = x$ by 6.4.1.

ie; $x - Px = 0$

$\Rightarrow (I - P)x = 0$

$\Rightarrow x \in \ker Q$

So ImP $\subseteq$ kerQ.

Now $x \in \ker Q \Rightarrow (I - P)x = 0$

$\Rightarrow Px = x$

$\Rightarrow x \in$ ImP.

$\therefore$ImP = kerQ.

Next, since Q is a projection ImQ = ker (I - Q) = ker (I - (I - P)) = ker P.

**Proposition 6.4.3:** Let $E_1$ = ImP and $E_2$ = kerP. Then $E_1+E_2 = E$ and

$E_1 \cap E_2 = \{0\}$.

ie: $E_1 + E_2$ is a direct sum and E is a direct decomposition on $E_1$ and $E_2$.

Proof : $\forall x \in E$, we have $x = Px + (I - P)x$. Note that $Px \in ImP$ and $(I - P)x \in kerP$. Now, let $x \in E_1 \cap E_2$ ie; $x \in ImP$ and $x \in \ker P$.

ie; $x \in kerQ = \ker (I - P)$ and $x \in kerP$.

$\therefore (I - P)x = 0$ and $Px = 0$

$\therefore Px = (1 - P)x = 0$.

ie; $x = 0$

So $E_1 \cap E_2 = 0$

Note: Combining these properties, we call P the projection onto $E_1$ parallel to $E_2$.

**Proposition 6.4.4**: Let $T : E \to E$ be any linear operator, $E_1 + E_2 = E$ and let P be the projection onto $E_1$ parallel to $E_2$. Then $PT = TP$ iff $E_1$ and $E_2$ are invarient subspaces of T.

Proof: Assume that $PT = TP$

Let $x \in E_1$. Then $Tx = T(Px) = TP(x) = PT(x) = P(Tx) \in Ex$.

$\therefore E_1$ is an invarient subspace of T.

Conversely, let both $E_1$ and $E_2$ be invarient subspaces of T. Let $x \in E$.

Then $x = x_1 + x_2$, where $x_1 \in E_1$ and $x_2 \in E_2$. Note that $P(x_1) = 0$ and $P(x_2) = 0$. Now $T(Px) = T(Px)$.

$$= T(P(x_1) + P(x_2)).$$

$$= T(x_1 + 0) = T(x_1).$$

Next, $PT(x) = P(Tx)$

$= P(T(x_1) + T(x_2))$

$= P(T(x_1) + P(T(x_2))$

$= T(x_1) + 0$

$= T(x_1)$.

$\therefore PT = TP$

Orthoprojections

A projection P in a Hilbert space is called an orthoprojection if $P = P^*$.

ie; $(Px, Py)\ \forall x \in H$.

**Proposition 6.4.5:** If P is an ortho projection then Im $P \perp$ ker

Proof: Let $x, y, \in H$.

Then $(Px, (I - P)y = (x, P(I - P)y)$

$= (x, (P - P^2)y)$

$= (x, (P - P)y)$

$= (x, 0y)$

$= 0$

$\therefore ImP \perp Im(I - P)$

ie; $ImP \perp kerP$.

Proposition 6.4.6: If P is a projection and $Imp \perp kerP$ then $P = P^*$.

Proof: Let $x, y \in H$. Then we have $x = x_1 + x_2$ and $y = y_1 + y_2$ where $x_1, y_1 \in E_1$ and $x_2, y_2 \in E_2$. Given that $E_1 \perp E_2$.

Now $(Px, y) = (P(x_1 + x_2), y_1 + y_2)$

$= (Px_1 + Px_2, y_1 + y_2)$

$= (x_1 + 0, y_1 + y_2)$

$= (x_1, y_1 + y_2)$

$= (x_1, y_1) + (x_1, y_2)$

$= (x_1, y_1) + 0$

$= (x_1, y_1)$.

Next $(x, Py) = (x_1 + x_2), P(y_1 + y_2)$

$= (x_1 + x_2, Py_1 + Py_2)$

$= (x_1 + x_2, y_1 + 0)$

$= (x_1, x_2, y_1)$

$= (x_1, y_1) + (x_2, y_1)$

$= (x_1, y_1)$.

**Proposition 6.4.7**: Every projection satisfies $0 \leq P \leq I$.

Proof: $(Px, x) = (P^2 x, y) = (Px, Px) = \|Px\|^2 \geq 0$

By 6.4.3, we have $x = Px + (I - Px)$, where $Px \in E_1$ and $(I - P)x \in E_2$.

$\|x\|^2 = \|Px\|^2 + \|(I - P)x\|^2$, (since $E_1 \perp E_2$)

$\therefore \|Px\|^2 = \|x\|^2 - \|(I - P)x\|^2$

$\leq \|x\|^2$

ie, $\langle Px, Px\rangle \le \langle \mathrm{x}, \mathrm{x}\rangle$

ie, $\langle x, P^2 x\rangle \le \langle x, x\rangle$

ie, $\langle Px, Px\rangle \le \langle \mathrm{Ix}, \mathrm{x}\rangle$

Hence we get $0 \le P \le \mathrm{I}$

*Let $\mathrm{P}_1$, $\mathrm{P}_2$ be two Ortho-projections. Let
$E_1 = \mathrm{Im} P_1$ *and* $\mathrm{E}_2 = \mathrm{Im} P_2$.

**Proposition 6.4.8:**

If
$P_1 P_2 = 0$ then $\mathrm{P}_2 P_1 = 0, E_1 \perp \mathrm{E}_2$ *and* $\mathrm{P}_1 + P_2$ is an orthoprojection on to $\mathrm{E}_1 \oplus E_2$

Proof: We have $P_2 P_1 = (P_1 P_2)^* = 0^* = 0$

$\forall x, y \in \mathrm{H}$ we have $\langle P_1 x, P_2 y\rangle = \langle P_2 P_1 x, y\rangle = \langle 0, y\rangle = 0.$

$\therefore E_1 \perp E_2$. It is clear that
$(P_1 + P_2)^2 = P_2$ *and* $(\mathrm{P}_1 + P_2)^* = P_1 + P_2$.

Next we proof $\mathrm{Im}(P_1 + P_2) = \mathrm{E}_1 + E_2$.

Let $\gamma \in \mathrm{I\,m} P_1 + P_2$. *So* $\exists\ \mathrm{x} \in \mathrm{H}$ such that $(\mathrm{P}_1 + P_2)x = y.$

*ie.* $P_1 x + P_2 x = y.$

Note that $P_1 x \in E_1$ and $\mathrm{P}_2 x \in E_2$.

$\therefore y \in E_1 + E_2$. *Thus* $\mathrm{Im}(P_1 + P_2) \subseteq E_1 + E_2$.

Conversely let $y \in E_1 + E_2$.

Then $y = y_1 + y_2$ where $\mathrm{y}_1 \in E_1$ *and* $\mathrm{y}_2 \in E_2$.

$\therefore \exists x_1, x_2 \in H$ such that $P_1 x_1 x_1 = y_1$ and $P_2 x_2 = y_2$.

Now $P_1 y_1 = y_1 \left[\because P_1 x_1 = P_1(P_1 x_1) = P_1 x_1 = y_1\right]$ and $P_2 y_2 = y_2$

We have $(P_1 + P_2)(y_1 + y_2) = P_1 y_2 + P_2 y_1 + P_2 y_2$

$= y_1 + P_1 P_2 x_2 + P_2 P_1 x_1 + y_2$

$= y_1 + 0 + 0 + y_2$

$= y_1 + y_2$

$= y$

$\therefore y \in \text{Im} P_1 + P_2$. Thus $E_1 + E_2 \subseteq \text{Im}(P_1 + P_2)$.

Since $E_1 \cap E_2 = \{0\}$ we get that $\text{Im}(P_1 + P_2) = E_1 \otimes E_2$.

**Proposition 6.4.9:**

If $P_1 P_2 = P_2 P_1 = P$ then P is an orthoprojection and E $= \text{Im} P = E_1 \cap E_2$.

Proof: Since $P_1$ and $P_2$ are equal to $Id / E_1 \cap E_2$ when restricted to $E_1 \cap E_2$ we get that $E_1 \cap E_2 \subseteq E$.

Also, since $P = P_1 P_2$ we get $P x \in E_1$ and since $P = P_2 P_1$ we get $P x \in E_2$. So $E \subseteq E_1 \cap E_2$.

**Proposition 6.4.10:**

If $P_1 P_2 = P_2 P_1 = P_1$ then $E_1 \to E_2$ and $P_1 \le P_2$.

Moreover, $P_1 \le P_2$ implies $P_1 P_2 = P_1$ and therefore $E_1 \to E_2$.

Proof: Assume $P_1 P_2 = P_1$. Then we have $(P_1 P_2)^* = P_1^*$

ie, $P_1^* = P_1$

ie, $P_2P_1 = P_1$.

First we prove $E_1 \to E_2$.

Let $y \in E_1$. Then $\exists x \in H$ 'such that $P_1 x = y$.

$\therefore$ we get $(P_2.P_1)x = y$

ie, $P_2(P_1)x = y$

$\therefore y \in E_2$.

Thus $E_1 \to E_2$.

Next, we have $(P_2 - P_1)^2 = P_2^2 - 2P_2P_1 + P_1^2$ (since $P_2P_1 = P_2P_1$)

$$= P_2 - 2P_1 - P_1$$

$$= P_2 - P_1.$$

$\therefore P_2 - P_1 \geq 0, ie, P_1 \leq P_2$.

Now, we assume that $P_1 \leq P_2$.

$\forall x \in H$ we have $\|P_1(1 - P_2)x\|^2 = \langle P_1(I - P_2)x, P_1(I - P_2)x \rangle$

$$= \langle P_1^2 (I - P_2)x, (I - P_2)x \rangle$$

$$= \langle P_1(I - P_2)x, (I - P_2)x \rangle$$

$$\leq \langle P_2(I - P_2)x, P_1(I - P_2)x \rangle \text{ (since } P_1 \leq P_2)$$

$$= \langle (P_2 - P_2^2)x, (I - P_2)x \rangle$$

$$= \langle (P_2 - P_2)x, (I - P_2)x \rangle$$

$$= \langle 0, (I - P_2)x \rangle$$

$\therefore$ we get $P_1(I - P_2)x = 0, \forall x \in H$..

ie, $P_1 x = P_1 P_2 x, \forall x \in H.$

$\therefore$ *we* get $P_1 P_2 = P_1.$

**Result:** Let $(P_n)$ be a sequence of orthoprojections converging strongly to an operator P. Then is an orthoprojection.

Proof: Given $P_n x \to Px \forall x \in H.$ Since P is a strong limit of self adjoint operators, P is also self adjoint.

Let

$$\left(x, y \in H. \text{Then } \langle \mathrm{Px}, \mathrm{y} \rangle = \langle \lim \mathrm{P_n} x, y \rangle = \lim \langle \mathrm{P_n} x, y \rangle = \lim \langle x, p_n y \rangle\right)$$
$$= \langle x, \lim P_n y \rangle = \langle x, Py \rangle$$

Since all $P_n$ are orthoprojections we have
$0 \le P_n \le I$ (Proposition 6.4.7).

$\therefore \|P_n\| \le 1.$ $\left(by\ 6.3\ \text{Property (i)}\right)$

We note that $P_n Px \to PPx = P^2 x \forall x \in H.$

Now, $\left\|P^2 x - P_n^2 x\right\| \le \left\|P^2 x - P_n Px\right\| + \left\|P_n Px - P_n^2 x\right\|$

$$= \left\|P^2 x - P_n Px\right\| + \left\|P_n \left(Px - P_n x\right)\right\|$$

$$\le \left\|P^2 x - P_n Px\right\| + \|P_n\| \ \|[\mathrm{Px} - \mathrm{p_n} x]\|$$

$$\le \left\|P^2 x - P_n Px\right\| + \|P_n\| \ \left(P\mathrm{x} - \mathrm{P_n} x\right)||$$

$$\to 0.$$

$\therefore \lim \mathrm{P}_n^2 x = P^2 x, \forall x \in H.$

So, we have $P^2x = \lim P_n^2 x = \lim P_n x = Px, \forall \in H$. Hence we get that *P* is an orthoprojection.

**Result:** Let $(P_n)$ be a decreasing sequence of orthoprojections converging strongly to an operator $P_1H_k$=I*mP*.

Then $H_0 = \bigcap_k Hk$

Proof: By the above result *P* is an orthoprojection.

We have $P \le P_k \forall k$. So $\text{Im}P \subseteq \text{Im}P_k \forall k$. (by Proposition: 6.4.7)

ie $H_0 \subseteq H_k \forall k$. $\therefore H_0 \subseteq \bigcap_k Hk$

Conversely, for any $x \in \bigcap_k H_k$, we have $x \in \text{Im}\, P_k \forall k$.

$\therefore Px = \lim_k P_k x = \lim_k x = x$ (since $x \in P_k x = x$)

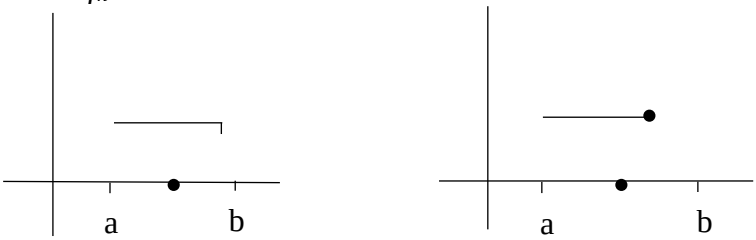
Hence $x \in \text{Im}P = H_0$. $\therefore \bigcap_k H_k \subseteq H_0$

Thus $H_0 = \bigcap_k H_k$

# 7. FUNCTIONS OF OPERATIONS; SPECIAL DECOMPOSITION

Let a, b, m, M $\in \mathbb{R}$ such that a < m ≤ M <b and let a be an operator satisfying mI ≤ A ≤ MI.

Let K[a,b] be the set of piecewise continuous bounded functions which are monotone decreasing limits of continuous functions. For a decreasing sequence $f_n$ converging to *f*, we write $f_n \searrow f$.




The first function in K; the second function not in K. Such functions are semi continuous from above. i.e., for all $a_0 \in \mathbb{R}$, all sets of the form t:x(t)≥ $a_0$ are closed sets.

**Lemma 7.0.1:** Let $\phi(t) \in K[a,b]$. Then there exists a sequence of polynomials $P_n(t) \searrow \phi(t)$ as $n \to \infty$ for all $t \in [a,b]$

**Proof:** By the definition of K[a,b], there exist $\phi_n(t) + 3/2^{n+2}$), we obtain that for all n there exists the polynomial $P_n(t)$ such that.

$$\left| P_n(t) - \left( \varphi_n(t) + \frac{3}{2^n + 2} \right) \right| \leq \frac{1}{2^{n+2}}$$

Then $P_{n+1}(t) \leq \varphi_{n+1}(t) + \frac{1}{2^{n+1}} \leq \varphi_n(t) + \frac{1}{2^{n+1}} \leq P_n(t)$

$\therefore$ we have $\frac{-1}{2^{n+2}} \leq P_n(t) - \varphi_n(t) - \left| \frac{3}{2^{n+2}} \right| \leq \frac{1}{2^{n+2}}$

$\therefore$ we get $P_n(t) \leq + \varphi_n(t) + \frac{3}{2^{n+2}} + \frac{1}{2^{n+2}} = \varphi_n(t) + \frac{1}{2^n}$ and

$$\varphi_n(t) \leq P_n(t) - \frac{3}{2^{n+2}} + \frac{1}{2^{n+2}} = P_n(t) - \frac{1}{2^{n+1}}$$

So, $P_{n+1}(t) \leq \varphi_{n+1}(t) + \frac{1}{2^{n+1}} \leq \varphi_n(t) + \frac{1}{2^{n+1}} \leq P_n(t)$

Thus, $P_n(t)$ is non increasing and obviously it converges to $\phi(t)$.

$$\left( \therefore \lim P_{n+1}(t) \leq \lim \left( (\varphi_{n+1}(t) + \frac{1}{2^{n+1}} \right) \leq \lim P_n(t) \right.$$

$$\left. \text{So} \therefore \lim P_n(t) \leq \varphi(t) \leq \lim P_n(t). \right)$$

**Definition 7.0.2:** (Defining $\phi(A)$). *Let* $Pn(t) \searrow \phi(t)$, for all $t \in [m,M]$. Then the decreasing sequence $P_1(A) \geq P_2(A) \geq \ldots P_n(A) \geq P_{n+1}(A) \geq \ldots$ is bounded. (since $\varphi(t) \in K[a,b]$), it is bounded $\therefore$ there exist some number T such that $\varphi(t) \geq -T$. Since $\{P_n(t)\}$ is decreasing and lim $P_n(t) = \phi(t)$, we get $P_n(t) \geq -T$. *i.e., $P_n(t)+T \geq 0$. So by 6.3.3 $P_n(A)+TI \geq 0$. i.e; $P_n(A) \geq -TI$.* Being a polynomial it is clear that $P_1(t) \leq K$ for some K. So again by 6.3.3 $P_1(A) \leq KI$. Thus $P_n(A) \leq KI$). So by theorem 6.3.2, the strong limit of $P_n(A)$ exists and we call it $\phi(A)$.

**Lemma 7.0.3:** Let $Q_n(t)$ and $P_n(t)$ be sequence of polynominals. Assume that for all $t \in [m,M]$, $Q_n(t) \searrow \psi(t) \in K$ and $P_n(t) \searrow \phi(t) \in K$. Let $\psi(t) \leq \phi(t)$ for all $t \in [m,M]$. Then lim $Q_n(A) = B_1 \leq B_2 = \lim P_n(A)$.

[As a consequence, if

$\psi(t) = \varphi(t)$, then $B_1 \leq B_2$ and $B_2 \leq B_1$.

This implies $B_1 = B_2 = \varphi(A)$

and the limit operator does not depend on a particular choice of an arbitrary sequence of polynomials point wise convergent to $\phi(A).]\,\varphi(t)$. So it justifies the definition of $\phi(A)$.

**Proof:** For any integer n and for every t $\in$ [m,M] there exists $N_o$ (t) such that for every $N \geq N_o(t), Q_N(T) < \psi(t) + \frac{1}{n}.$

We have $\psi(t) + \frac{1}{n} \leq \phi(t) + \frac{1}{n} \leq P_n(t) + \frac{1}{n}.$

$\therefore$ we get $Q_N(t) < P_n(t) + \frac{1}{n}$ (*)

Since $Q_N$(t) is continuous, $\exists$ an open interval I (t) around t where (*) is also satisfied for $N=N_o(t)$ and, consequently, for every $N>N_o(t)$. So we have a covering of [m, M] by open intervals.

$\therefore$ by Heine-Borel theorem, $\exists$ a finite sub covering $\{I(t_i)\}_{i=1}$

Let $N_o = \max N_o(t_i)$.

Then for all n and for every $N>N_o$, we have

$Q_N(t) < p_n(t) + \frac{1}{n}$ for every t$\in$[m,M].

$\therefore$ we get $Q_N(A) \leq p_n(A) + \frac{1}{n} I.$

Now by letting N$\to \infty$, we get $B_1 \leq B_2.\}$ By letting $N \to \infty$, we get $B1 \leq P(n)(A) + \frac{1}{n} I$ so far n is assumed fixed.

**Note:** For fixed operator A, the correspondence $\phi \in K \mapsto \phi(A) \in L(H)$ is well defined.

**Exercise 7.0.4:** Let $\phi_n(t) \searrow \phi(t)$. Then $\phi_n(A) \to \phi(A)$ strongly.

**Remark:** If A is a self-adjoint operator and $P_n(t)$ is a real polynomial, then $P_n(A)$ is also self-adjoint.

[Let $P_n(t)=a_0+a_1t+....+a_nt^n$, where $a_0, ... a_n \in \mathbb{R}$

$P_n(A)=a_0I+a_1A+...a_nA^n$

Then $P_n(A)^* = \left(a_0 I + a_1 A + ... + a_n A^n\right)^*$

$= a_0 I^* + a_1 A^* + ... + a_n \left(A^n\right)^*$

$= a_0 I + a_1 A + ... + a_n A^n = P_n(A)]$

$\therefore$ for every function $\phi \in k$, the operator $\phi(A)$ is self - adjoint .

$[\forall\ x, y \in H, \langle \phi(A)x, y\rangle = \langle \lim p_n(A)x, y\rangle = \lim \langle P_n(A)x, y\rangle$

$= \lim \langle x, P_n(A)y\rangle = \langle x, \lim P_n(A)y\rangle$

$= \langle x, \phi(A)y\rangle$

**Proof of 7.04:** As in the proof of 7.0.1, for all n, there exists polynomial $P_n(t)$ such that

$P_{n+1} \le \phi_n(t) + \frac{1}{2^n + 1} \le P_n(t)$. So we have $P_n(t) \searrow \phi(t)$ and

$\phi(A) = \text{strong limit of } P_n(A)$

$P_{n+1}(A) \le \phi_n(A) + \frac{1}{2^n + 1} I \le P_n(A)$. As $n \to \infty$, we get

$p_{n+1}(A) \le \phi n(A) + \frac{1}{2^{n+1}} I$

$\lim \phi_n(A) = \phi(A)$. Consider the correspondence $K \to L(H)$

$\phi \mapsto \phi(A)$.

Next we establish linear, multiplicative and other type properties of this correspondence.

**(i)** Let $\phi_1, \phi_2 \in K$. Then $\phi_1 + \phi_2 \mapsto \phi_1(A) + \phi_2(A)$.

ie. $(\phi_1 + \phi_2)(A) \to \phi_1(A) + \phi_2(A)$.

Proof: Let $P_n^{(1)} \searrow \phi_1$ and $P_n^{(2)} \searrow \phi_2$.

Then $P_n^{(1)} + P_n^{(2)} \searrow \phi_1 + \phi_2$.

$\therefore$ we have $(\varphi_1 + \varphi_2)(A) = \lim \left[\lim P_n^{(1)} + P_n^{(2)}\right]$

$= \lim P_n^{(1)}(A) + \lim P_n^{(2)}(A)$

$= \phi_1(A) + \phi_2(A)$.

**(ii)** for c > 0, we have $C\phi_1(A)$. (if $c < 0$ then $c\phi_1$ may not belong to the class K).

**(iii)** Let $\phi_1, \phi_2 \geq 0$. Then $(\phi_1.\phi_2)(A)$. $\phi_2(A)$. (we assumed $\phi_1, \phi_2 \geq 0$ because otherwise $\phi_1.\phi_2$ may not belong to the class K).

Proof: Take $P_n^{(1)} \searrow \phi_1$ and $P_n^{(2)} \searrow \phi_2$.

Then $P_n^{(1)}.P_n^{(2)} \searrow \phi_1.\phi_2$.

$\therefore$ we get $(\varphi_1.\varphi_2)(A) = \varphi_1(A).\varphi_2(A)$.

(Proof: Similar to the proof of 7.0.3).

**Note:** W shall now extend this correspondence.

Consider the linear classes of functions K-K of the form $\psi = f - \Phi$ where $f, \Phi \in K$. Then we denote $\psi(A) = f(A) - \Phi(A)$.

**Claim:** $\Psi(A)$ is well defined ie., its definition is independent of the presentation of as a difference $f - \Phi$.

Proof: Assume that $\Psi = f_1 - \Phi_1 = f_2 - \Phi_2$.

Then $f_1 - \Phi_2 = -\Phi_1 + f_2$.

Therefore we get $(f_1 + \Phi_2)(A) = (\Phi_1 + f_2)(A)$

$f_1(A) + \Phi_2(A) = \Phi_1(A) + f_2(A)$

$f_1(A) - \Phi_1(A) = f_2(A) - \Phi_2(A)$. So our claim holds.

**Note:** We may take any constant c in property (ii) and we do not need to assume. $\Phi_1, \Phi_2 \geq 0$ in property (iii)

**Proof:** (find 9
$c_1 \geq 0$ such that $c\Phi + c_1 \geq 0$ and $c\Phi + c_1 \in K$ now, we have

$c\Phi = (c\phi + c_1$ -$c_1 \in$ (K -K) and define (cΦ) (A) using this identity.

Now let us show the second statement. Let $c_1$, $c_2$ be constants such that $\Phi_1 + c_1 \geq 0$ and $\Phi_2 + c_2 \geq 0$. Then we have,

$\Phi_1\Phi_2 = (\Phi_1 + c_1)(\Phi_1 + c_1) - (c_1\Phi_2 + c_2\Phi_1 + c_1c_2) \in K - K.$

The operator $(\Phi_1\Phi_2)$ (A)is defined through this identity and equals $\Phi_1(A)\Phi_2(A)$.

**Remark:** We may also extend this correspondence on complex valued functions: for
$g(t) = \Phi(t) + i\Psi(t)$ with $\Phi(t), \Psi(t) \in K - K$ we define

$g(A) = \Phi(A) + i\Psi(A)$.

### 7.1 Spectral Decomposition

Let A be an operator on a Hilbert space H, such that
$mI \leq A \leq MI$, where $a < m \leq M < b$.

Consider the function $e_\lambda(t)$ $= 1$, for $t \le \lambda$ ; $= 0$, for $t > \lambda$ where $\lambda \in \mathbb{R}$

obviously $e_\lambda(t) \in K[a,b]$

Define $E_\lambda = e_\lambda(A)$.

i) $E_\lambda$ is an orthoprojection; $E_\lambda = 0$ for $a < \lambda < m$ and $E_\lambda = I$ for $\lambda \ge M$.

**Proof:** We have $(e_\lambda . e_\lambda)(t) = e_\lambda(t)$, $(e_\lambda(t)).(e_\lambda(t)) = e_\lambda(t)$.

ie., $(e_\lambda . e_\lambda)(t) = e_\lambda(t)$

$\therefore (e_\lambda . e_\lambda)(A) = e_\lambda(A) =$

i.e., $e_\lambda(A).e_\lambda(A) = e_\lambda(A)$

i.e., $E_\lambda . E = E_\lambda$

So, $E_\lambda$. is a projection

Since $e_\lambda(t)$ is a real valued function, we get that $E_\lambda$ is symmetric.

$\therefore E_\lambda$ is an orthoprojection

Now, we prove $E_\lambda = 0$ for $a < \lambda < m$ and $E_\lambda = I$ for $\lambda \ge M$.

If $a < \lambda < m$, then $e_\lambda = 0$, the zero function on [m,M].

$\therefore E_\lambda = e_\lambda(A) = 1$

**ii** $E_\lambda$ is continuous from the right w.r.t. $\lambda$ in the strong sense.

**Proof:** Let $\phi_n(t)$ be the sequence of continuous functions such that $\phi_n(t) \ge e_{\lambda+\frac{1}{n}}(t)$ *and* $\phi_n(t) \searrow e_\lambda(t) \le \searrow e_\lambda(t)$

Then by exercise 7.0.4, we get

$$\phi_n(A) \ge E_{\lambda+\frac{1}{n}} \ge E_{\lambda+\alpha_n} \text{ for } \frac{1}{n} \ge \alpha_n \ge 0.$$

As n $\to\infty$, $\phi_n(A) \searrow E_\lambda$

So, $E_{\lambda+\alpha_n} \to E_\lambda$ as $\alpha_n \to 0$

$E\lambda$($\alpha_n$ not required)

**iii)** $E_\lambda E_\mu = E_\lambda$ for $\lambda<\mu$

Proof: we note that $e_\lambda(t)e_\mu(t)$, for $\lambda<\mu$

$e_\lambda(t).\ e\mu(t) = e_\lambda(t)$

We have $e_\lambda(t) = \begin{cases} 1 & t \le \lambda \\ 0 & t > \lambda \end{cases}$ and $e_\mu(t) = \begin{cases} 1 & t \le \mu \\ 0 & t > \mu \end{cases}$

now $e_\lambda(t)\, e_\mu(t) = \begin{cases} 1 & t \le \lambda \\ 0 & \lambda < t \le \mu \\ 0 & t > \mu \end{cases} = \begin{cases} 1 & t \le \lambda \\ 0 & t > \lambda \end{cases} = e_\lambda(t)$

$\therefore$ we get $E_\lambda E_\mu = E_\lambda$ for $\lambda < \mu$

**Note:** Property (iii) may also written as $E_\lambda \le E_\mu$ for $\lambda<\mu$

A family { $E_\lambda$}with properties (i), (ii) and(iii) is called a 'spectral family' or a "decomposition of identity."

iv) $E_\lambda A = AE_\lambda$ since $E_\lambda$ is a strong limit of Polynomials of A.

$\therefore$ Im $E_\lambda = \mu_\lambda$ is a invariant subspace of A (by 6.4.4).

### 7.1.a The main inequality

Let $\lambda_1 < \lambda_2$ Then

$$\lambda_1(e_{\lambda 2}(t) - e_{\lambda 1}(t)) \le t(e_{\lambda_2}(t)) - \le \lambda_2(e_{\lambda_2}(t) - e_{\lambda_1}(t))$$

Inserting A in place of 't' and using the correspondence between functions and operators, we get our main inequality

$\lambda_1(E_{\lambda 2}(t) - E_{\lambda 1}(t)) \le A(E_{\lambda 2}(t) - E_{\lambda 1}(t)) \le_2 E_{\lambda 2}(t) - E_{\lambda 1}(t)) - - - (*)$

Some partial cases of this inequality are of special interest.

Taking

$\lambda_1 < m$ and $\lambda_2 = \lambda_1$, we have $AE \le \lambda E_\lambda$ since $E_{\lambda 1} = 0$ for $\lambda_1 < m$.

Taking $\lambda_1 < m = and\ \lambda_2 > M$, we have

$\lambda(I - E_\lambda) \le A(I - E_\lambda)$, since $E_{\lambda 2} = I$ for $\lambda_2 > M$

*or* $(A\text{-}\lambda I)E_\lambda \le A\text{-}\lambda I$

**Remark:** If $\lambda_1 < \lambda_2$, then we observe that $E_{\lambda_1 . \lambda_2} = E_{\lambda_2} - E_{\lambda_1}$ is an orthoprojection.

**Proof:**

$(E_{\lambda_2} - E_{\lambda_1})^2 = E_{\lambda_2}^{\ 2} - E_{\lambda_1}E_{\lambda_2} + E_{\lambda_1}^{\ 2} = E_{\lambda_2} - E_{\lambda_1} - E_{\lambda_1} + E_{\lambda_1} = E_{\lambda_2} - E_{\lambda_1}$

Also $(E_{\lambda_2} - E_{\lambda_1})^* = E_{\lambda_2} - E_{\lambda_1}$

$\therefore E_{\lambda_2} - E_{\lambda_1}$ is an orthoprojection

**Note:** Since $E_{\lambda_1 \lambda_1}$ commutes with A, the subspace $H_{\lambda_1 \lambda_2} = \mathrm{Im} E_{\lambda_1 \lambda_2}$ is an invariant subspace of A.

For $x \in H_{\lambda_1 \lambda_2}$ we have $\lambda_1 I H_{\lambda_1 \lambda_2} \le A / H_{\lambda_2} I H_{\lambda_1 \lambda_2}$

$\lambda_1 . I._{H_{\lambda_1 \lambda_2}} \le A / H_{\lambda_1 \lambda_2} \le \lambda_2 I_{H_{\lambda_1 \lambda_2}}$

i.e, $< \lambda_1 E_{\lambda_1 \lambda_2} x, x > \le < A E_{\lambda_1 \lambda_2} x, x > \le < \lambda_2 E_{\lambda_1 \lambda_2} x, x > \forall x \in H$

Now, let $x \in H_{\lambda_1 \lambda_2}$ Then $E_{\lambda_1 \lambda_2} x = x$ since $E_{\lambda_1 \lambda_2}$ is a projection

$\therefore$ we have

$< \lambda_1 x, x > \le < Ax, x > \le < \lambda_2 x, x > \forall x \in H_{\lambda_1 \lambda_2}$

So $\lambda_1 IH_{\lambda_1\lambda_2} \le A/H_{\lambda_1\lambda_2} \le \lambda_2 I_{H\lambda_1\lambda_2}$ ]

$\therefore$ for all $\lambda \in [\lambda_1\lambda_2]$*we* have

$-\varepsilon I_{H_{\lambda_1\lambda_2}} \le$ (A - $\lambda$1)/H$_{\lambda_1\lambda_2}$ $\le \in I_{H_{\lambda_1\lambda_2}}$ where $\in = \lambda_2 - \lambda_1$

[On $_{H\lambda_1\lambda_2}$ we have $\lambda_1 I \le A$

This implies $\lambda_1 I - \lambda_2 I \le A - \lambda_2 I \le A - \lambda I$, since $-\lambda_2 \le -\lambda$

So $(\lambda_1 - \lambda_2)\, I \le A - \lambda I$

i.e. $-\mathcal{E} I \le (A - \lambda I)$

Also on $H_{\lambda_1\lambda_2}$, we have $A \le \lambda_2 I$

This gives $\lambda_2 I - \lambda_1 I \ge$ A– $\lambda_1 I \ge$ A–$\lambda$I, Since $-\lambda_1 \ge -\lambda$

So $(\lambda_2 \lambda_1)\, I \ge A - \lambda I$

i.e. $\varepsilon I \ge A - \lambda I$]

$\mathcal{E} I \le A$ - $\lambda I$

By property (i) of 6.3.a we get || A–$\lambda$I|| $_{H_{\lambda_1\lambda_2}} \le \varepsilon$

[We have – $I \le \frac{1}{\varepsilon}(A - \lambda I) \le I$ on $H_{\lambda_1\lambda_2}$

$\therefore \quad \left\| \frac{1}{\in}(A - \lambda I) \right\| \le 1$

So $\|(A - \lambda I)\| \le \in$]

This means that the operator A is close to a constant operator on the sub space $H_{\lambda_1\lambda_2}$. However, we do not know that $H_{\lambda_1\lambda_2} \ne \{0\}$, then we say that $\lambda$is "almost" an eigen value and every non-zero element of $H_{\lambda_1\lambda_2}$ is almost an eigen vector.

### 7.1.b Construction of the spectral integral

We shall build an integral, which is called the "Spectral integral." Consider the partition of

$(a,b): a < \lambda_0 < m \le \lambda_1 < .... < \lambda_{n-1} \le M < \lambda_n < b$ with norm of the partition $\Delta = \max\left|\lambda_{i+1} - \lambda_i\right| < \in$. Choose any $\mu \in \left[\lambda_i, \lambda_{i+1}\right]$

Adding (*) we get

$$= \sum_{k=0}^{n-1} \lambda_k (E_{\lambda_{k+1}} - E_{\lambda_k})$$

$$\le A \sum_{k=0}^{n-1} \lambda_k (E_{\lambda_{k+1}} - E_{\lambda_k})$$

$$= A \le \sum_{k=0}^{n-1} \lambda_{k+1} (E_{\lambda_{k+1}} - E_{\lambda_k})$$

$$\left[\sum_{k=0}^{n-1} \lambda_k (E_{\lambda_{k+1}} - E_{\lambda_k}) = E_{\lambda_1} - E_{\lambda_0} + E_{\lambda_2} - E_{\lambda_1} + E_{\lambda_3} - E_{\lambda_2} + ... + E_{\lambda_n} - E_{\lambda_{n-1}}\right.$$

$$= E_{\lambda_1} - 0 + E_{\lambda_2} - E_{\lambda_1} + E_{\lambda_3} - E_{\lambda_2} + ... + I - E_{\lambda_{n-1}}$$

$$\left. = \mathrm{I}\right]$$

Consider the operator $\sum_{k=0}^{n-1} \mu_k \ (E_{\lambda_{k+1}} - E_{\lambda_k})$

Since $-\varepsilon \le \lambda_k - \mu_k$ and $\lambda_{k+1} - \mu_k \le \varepsilon,$ we have

$$-\varepsilon I = -\varepsilon \sum_{k=0}^{n-1} (E_{\lambda_{k+1}} - E_{\lambda_k})$$

$$\le \sum_{k=0}^{n-1} (\lambda_k - \frac{\mu k}{\mu_k})(E_{\lambda_{k+1}} - E_{\lambda_k})$$

$$= \sum_{k=0}^{n-1} \lambda_k (E_{\lambda_{k+1}} - E_{\lambda_k}) - \sum_{k=0}^{n-1} \mu_k (E_{\lambda_{k+1}} - E_{\lambda_k})$$

$$= A - \sum_{k=0}^{n-1} \mu_k \ (E_{\lambda_{k+1}} - E_{\lambda_k})$$

$$\leq A - \sum_{k=0}^{n-1} \mu_k \ (E_{\lambda_{k+1}} - E_{\lambda_k})$$

$$\leq \sum_{k=0}^{n-1} (\lambda_{k+1} \ (E_{\lambda_{k+1}} - E_{\lambda_k}) - \leq \sum_{k=0}^{n-1} \mu_k \ (E_{\lambda_{k+1}} - E_{\lambda_k})$$

$$\sum_{k=0}^{n-1} (\lambda_{k+1} - \mu_k)(E_{\lambda_{k+1}} - E_{\lambda_k})$$

$$\leq \sum_{k=0}^{n-1} \varepsilon (E_{\lambda_{k+1}} - E_{\lambda_k})$$

$$= \varepsilon I$$

So we have $-\varepsilon I \leq A - \sum_{k=0}^{n-1} \mu_k (E_{\lambda_{k+1}} - E_{\lambda_k}) \leq \varepsilon I$

This gives $\left\| A - \sum_{k=0}^{n-1} \mu k (E_{\lambda_{k+1}} - E_{\lambda_k} \right\| \leq \varepsilon$ by property (i) of 6.3. a, for any partition of the interval, where the partitions norm is less than ε, and any choice of $\mu_k$ inside the interval of the partition.

Then there is a limit (in the operator norm) as $\varepsilon \to 0$ and the name for the limit is "integral". So we define the notation of the spectral integral.

$$A = \int_a^b \lambda \, \mathrm{dE}_\lambda$$

$$= \int_{m-0}^{M} \lambda \, \mathrm{dE}_\lambda$$

$$= \int_{\infty}^{\infty} \lambda \, \mathrm{dE}_\lambda$$

We may take any a < m as the lower integration point which we express by writing m-0. We may also take any b>M; but because of continuity from the right of $E_\lambda$, we may take the upper bound to be exactly M.

We note that $E_\lambda = I$ for $\lambda \geq M$ and $E_\lambda = 0$ for $\lambda < m$

and so we may formally write $= \int_{\infty}^{\infty} \lambda \, dE_{\lambda}$ since $dE_{\lambda}$ is identically zero outside [m, M].

## 7.2 HILBERT THEOREM ON THE SPECTRAL DECOMPOSITION OF SELF ADJOINT BOUNDED OPERATORS:

**Theorem 7.2.1**

For every self adjoint bounded operator on a Hilbert space H such that ml ≤ A≤ MI, there exists a family $\{E_{\lambda}\}, \lambda \in R$ of orthoprojections such that the following are true.

i) $E_{\lambda} = 0$ for $\lambda < m$ and $E_{\lambda} = I$ for $\lambda \geq M$

ii) $E_{\lambda+0} = E_{\lambda}$ is continuous from the right w.r.t $\lambda$ )

iii)
$E_{\lambda_1} \leq E_{\lambda_2}$ for $\lambda_1 < \lambda_2$ or equivalent ly $E_{\lambda_1} E_{\lambda_2} = E_{\lambda_1}$ for $\lambda_1 < \lambda_2$.
(properties (i) - (iii) means $\{E_{\lambda}\}$ is a spectral family).

iv) $A = \int_{-\infty}^{\infty} \lambda \, d \, E_{\lambda}$ with the integral converging in the operator norm.

v) $E_{\lambda}$ are strong limit of polynomials of A and therefore they commute with any operator B which commutes with A.

vi) $Ax^2 = \int \lambda^2 \, d \, <E_{\lambda} x, x>$

vii) A family $\{E_{\lambda}\}$ that satisfies (i) - iv) is unique.

### 9.1 Auxiliary Results

In a set M we define a distance to be a non-negative function $\rho$ *(x, y); x, y* $\in M$, which is called a metric and which

satisfies the following conditions.

1. $\rho(x, y) = 0$ if and only if x = y

2. $\rho(x,y()) = \rho(y, x)$

3. $\rho(x, z) \leq \rho(x, y) + \rho(yz)$

The set M with a metric $\rho$ is called a metric space.

A metric space M is called complete if for any sequence $\{ x_n \} \subseteq M$ such that $\rho(x_m, x_n) \to 0$ *as* m, $n \to \infty$ there exist a limit $x \in M$ i.e, $\lim_{n\to\infty} \rho(x_n, x) = 0$

For a subset $A \subseteq M$ is said to be we denote $A^0$, $\bar{A}$, $A^c$ the set of interior points, the closure, and the complement of A.

A subset $A \subseteq M$ is said to be nowhere dense if $\bar{A}$ does not contain any interior points.

Result: if a set A is nowhere dense, then for any open ball $B \subseteq M$ there exists an open ball $B_1 \subseteq B$ such that $B_1 \cap A = \phi$

**Proof**: Let $B \subseteq M$ be an open ball.

claim : $B-\bar{A} \neq \phi$

pf:if possible, let $B-\bar{A} \neq \phi$. so B does not intersect woth $\bar{A}^c$

$B \subseteq \bar{A}$

therefore $\bar{A}$ contains interior points, which is contradiction.

So our claim holds.

The set $B - \bar{A}$ os open and hence there exists an open ball $B_1 \subseteq B - \bar{A}$.

For this ball, clearly $B_1 \cap A = \phi$.

Hence the result is proved.

**Remark**: The converse of the above result is also true.

That is, let A be a set in a metric space M. Assume that for any open ball $B \subseteq M$, there exist an open ball $B_1 \subseteq B$ such that $B_1 \cap B = \phi$. Then A is nowhere dense.

**Proof**: If possible assume that A is not nowhere dense. Then $\bar{A}$ has an interior point , say $x_0$.

Let $V_{x_0}$ be an open ball with centre at $x_0$ such that $V_{x_0} \subseteq \bar{A}$

Therefpore any open ball $B_1 \subseteq V_{x_0}$, we have $B_1 \cap A \neq \phi$,

which is a contradiction to our assumption.

**Definition**: A subset $A \subseteq M$ is said to be of *first category* if A is the union of countably many sets $A = \bigcup_i B_i$ and the $B_i$'s are nowhere dense for all i.

A set C which is not of first category is called a *set of second category*.

Thorem 9.1.1: (Baire-Hausdoff)

Every complete metric space M is of second category.

Proof: If possible, assume that M is of first category.

i.e; $M = \bigcup_{n=1}^{\infty} A_n$ where $A_n$ is nowhere dense for all $n \geq 1$.

Let $B_0 = D(x_0, 1)$ be a closed ball with radius 1 and centre $x_0$

Since $A_1$ is nowhere dense, there exist $\in_1 > 0$ and and a ball $B_1 = D(x_1, \in_1)$ such that $B_1 \subseteq B_0$ and $B_1 \cap A_1 = \phi$. We can

choose the ball $B_1$ to be closed and $\in_1 < 1/2$.

since $A_2$ is nowhere dense, there exist $0 < \in_2 < 1/3$ and a closed ball $B_2 = D(x_2, \in_2)$ such that $B_2 \subseteq B_1$ and $B_2 \cup A_2 = \phi$. Continuing like this, we get a sequence of closed balls

$B_n = D(x_n, \in_n)$ where $0 < \in_n < \frac{1}{n+1} + B_{n+1} \subseteq B_n$ and $B_n \cup A_3 = \phi$.

The sequence $(x_n)$ of centers is a Cauchy sequence, since for any m>n, $x_m \in B_n$ and hence $\rho(x_m, x_n) < 1/n+1 \to 0$ as $n \to \infty$.

Since M is complete, there exist $y_0 \in M$ such that $y_0 = \lim_{n\to\infty} x_n$

claim : $y_0 \in \bigcup_{n=1}^{\infty} B_n$

pf: We note that every ball $B_n$ contains all the points of the sequence $(x_k)$ except may be the points $x_1, x_2, x_3 ....., x_{n-1}$.

Therefore $y_0$ is a limit point of every closed ball $B_n$.

Hence $y_0 \in B_n$ for every n.

So $y_0 \in \bigcup_{n=1}^{\infty} B_n$ and our claim is proved.

Thus $y_0 \notin A_j$, for every j=1, 2, 3..... $M \neq \bigcup_{j=1}^{\infty} A_j$

This gives which is a contradiction.

**Definition**: Let X be Banach space. A set $K \subseteq X$ is called perfectly convex if and only if for every bounded sequence $(x_i)$ in K and for every sequence of reals $\alpha_i \geq 0$ such that $\Sigma i = 1$ $\alpha_i = 1$ we have that $\sum_{i=1}^{\infty} \alpha_i x_i \in K$

**Example:**

Let X=R and K=[0, 1]. Then K is perfectly convex.

pf: Let $x_1, x_2, x_3$ .... be a bounded sequence in K. Therefore $(0 \le x_n \le 1)$.

Let $\alpha_1, \alpha_2, \alpha_3$..... be a real sequence with $\alpha_n \ge 0$ and $\Sigma\alpha_n = 1$

Now, $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 +$ ..... $\le \alpha_1 + \alpha_2 +$..... = 1.

Therefore $\sum_{n=1}^{\infty} \alpha_n x_n \in K$

Therefore K is perfectly convex.

Note: Every perfectly convex set is convex.

Proof: Let K be a perfectly convex set. Let $x, y \in K$ and $0 \le \lambda \le 1$. Consider the bounded sequence x, y, x, y,..... in K and the real sequence $\lambda$, 1- $\lambda$, 0, 0, .....

Since K is perfectly convex, we get that $\lambda x + (1 - \lambda) y + 0x + 0y + ... \in K$.

i.e, $\lambda x + (1 - \lambda) y \in K$. Hence K is convex.

Remark: But a convex set in a Banach space need not be pefectly convex.

The following example illustrates the fact

**Example:** Consider the Banach space $l_2$. Let $e_n$ = (0,0,0, …..1, 0, …..) for n=1,2,3 …… Let A be the set of all finite convex combinations of $\{e_1, e_2, .....\}$

Then A is convex, but A is not perfectly convex.

**Remark**: $C_{00}$ is a convex subset of $l_1$, but $C_{00}$ is not perfectly convex.

**Proof**: Since $C_{00}$ is a subsapce of $l_1$, we get that $C_{00}$ is convex. As in the earlier note, we get that $C_{00}$ is not perfectly convex.

**Note**: A closed convex subset of a Banach space is perfectly convex.

pf : Let K be a closed convex set in a Banach space.

Let $(x_n)$ be a bounded sequence in K. Let $\alpha_n \geq 0$ be such that $\Sigma n=1^{\infty} \alpha_{n=1}$.

We want to show that $\sum_{k=1}^{\infty} \alpha_k X_k \in K$.

Let $S_n = \alpha_1 x_1 + \alpha_2 x_2 + \ldots\ldots + \alpha_n x_n + (\alpha_{n+1} + \alpha_{n+2} + \ldots\ldots) x_{n+1}$

Then clearly $S_n \in K$, since K is convex

Also we get, $S_n \to \sum_{k=1}^{\infty} \alpha_k x_k$. Since K is closed, we get that $\sum_{k=1}^{\infty} \alpha_k x_k \in K$

Therefor K is perfectly convex.

**Definition** : The *Kernal* of K (or the centre of K) is defined to be the set

$$\overset{c}{K} = \{x \in K; \text{for all } y \in X, \text{ there exist } \alpha > 0 \text{ with } \lambda y + (1-\lambda)x \in K, \text{for all } o \leq \lambda \leq \alpha\}$$

**Note :** Geometrically, $x \in \overset{c}{K}$ if and only if for any $y \in X$ atleast some part of the line segment joining x and y must belong to K.

Eg: If X = R and K= [0,1] then $\in \overset{c}{K}$ = (0, 1).

Example : Let X be a Banach space and D=B(0,r) be an open ball around 0. Then $\overset{c}{D} = D$. Proof: Let $x \in D$. Then $\exists \varepsilon > 0$ such that $B(x, \varepsilon) \subseteq D$, since D is open.

Let $y \in X$ is arbitrary, Choose

$\alpha = \dfrac{\varepsilon}{2\|y-x\|}$. *Then for any* $\lambda$ *with* $o \le \lambda \le \alpha$

We have $\|\lambda y + (1-\lambda)x - x\| = \|\lambda(y-x)\| = \lambda\|y-x\| \le \varepsilon/2 < \varepsilon.$

$\therefore \lambda y + (1-\lambda)x \in B(x,\varepsilon) \subseteq D.$

$\therefore x \in \overset{c}{D}$. *so we get* $D \subseteq \overset{c}{D}$

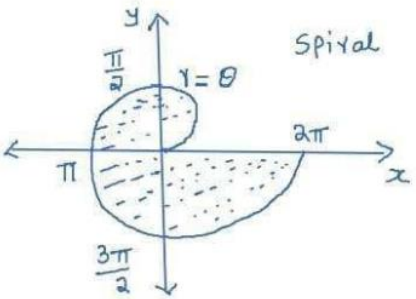
Since always $\overset{c}{D} \subseteq D$, *we get that* $D = \overset{c}{D}$

**Remarks :** It is clear that $\overset{0}{K} \subseteq \overset{c}{K}.$

But there may be points in the centre, which are not interior points.

For example, in $\Re^2$ *let* $K = \{(r,\theta); r \le \theta, 0 \le \theta \le 2\pi\}$ in polar coordinates.

Then $(0,0) \in \overset{c}{K}$, but $(0,0) \notin \overset{0}{K}$.




**Note:** For any

$x \in X$, *if we put* $k_x = k - x$ *we have* $(\overline{K_x}) = \overline{K} - x, \left(\overset{0}{K_x}\right) = \overset{0}{K} - x, \overset{c}{K}_x = \overset{c}{K} - x$

**Proof :**

a)

$y \in \overline{K} - x \Leftrightarrow y + x \in \overline{K} \Leftrightarrow y + x = z \in \overline{K} \Leftrightarrow y + x = z = \lim z_n \ with \ z_n \in K$

$$\Leftrightarrow y = \lim z_n - x = \frac{\lim (z_n - x), with\, z_n - x \in K_x}{\Leftrightarrow y \in \overline{K_{x.}}}$$

$Thus\left(\overline{K_x}\right) = \overline{K} - x$

b) Let

$y \in \left(\overset{0}{K}_x\right)$. Then definitely $y \in K_x$. i.e.; $y = v - x$, where $v \in K$

Moreover, there exist $\varepsilon > 0$ such that $U(Y,\varepsilon) \subseteq K_{x.}$

i.e; there exist $\varepsilon > 0$ such that for any z with $\|z - y\| < \varepsilon$, we have $z \in K_x$; i. e; $z = u - x$, with $u \in K$.

Next, we show that $U(v,\varepsilon) \subseteq K$. Let $u_D \in U(v,\varepsilon)$

i.e. $\|U_{D-v}\| < \varepsilon$ Taking $z = U_D - x$, we have

$\|z - y\| = \|u_D - x - v + x\| = \|u_D - v\| < \varepsilon$. This implies $U_D - x = u - x$

i. e; $U_D = u \in K$ . $\therefore$ we get $U(\nu,\varepsilon) \subseteq K$.

So $v \in \left(\overset{0}{K}\right)$ i. e; $y \in \left(\overset{0}{K}\right) - x$. Hence $\left(\overset{0}{K}_x\right) \subseteq \left(\overset{0}{K}\right) - x$

Conversecly, let $y(x) \in \left(\overset{0}{K}\right) - x$. $i.e; y = v - x, where\ v \in (\overset{0}{K})$.

This means that there exist $\varepsilon > 0$ such that for any u with $\|u - v\| < \varepsilon$, we have

$u \in K$.

we show that $U(y, \varepsilon) \subseteq K_x$, so that y becomes an interior point of $K_x$.

let $z \in U(y,\varepsilon) \subseteq K_{x,}$ so that y becomes an interior point of $K_x$

let $z \in U(y, \in)$.i.e; $\|y - z\| < \varepsilon$

$\therefore \|v - (z + x)\| = \|y + x - z - x\| = \|y - z\| < \varepsilon$

Hence $z + x \in K$.i.e; $z \in K_{x.}$ Thus $U(y,\varepsilon) \in K_x \therefore y \in \overset{0}{K}_x$

So $\left(\overset{0}{K}\right) - x \subseteq \left(\overset{0}{K}_x\right)$.

c) Finally, $z \in \overset{c}{K} - x \Leftrightarrow y = z + x \in \overset{c}{K} \Leftrightarrow$

for any
$u \in X$, *there exist* $\alpha > 0$ *such that* $\lambda u + (1-\lambda) y \in K$, *for* $0 \le \lambda \le \alpha \Leftrightarrow$
for any
$u \in X$, *there exist* $\alpha > 0$ *such that* $\lambda u + (1-\lambda) x \in K_x$, *for* $0 \le \lambda \le \alpha \Leftrightarrow$
for any
$u - x \in X$, *there exist* $\alpha > 0$ *such that* $\lambda (u - x) + (1-\lambda)(y - x) \in K_x$, *for* $o \le \lambda \le \alpha$
$\Leftrightarrow$ (For any $V \in X$, there exist $\alpha > 0$ such that $\lambda V + (1-\lambda)$ (y-x)
$\in K_\infty$, for $0 \le \lambda \le \alpha$) $\Leftrightarrow y - x \notin \overset{c}{K}_x \Leftrightarrow z \in \overset{c}{K}_{x.}$

**Note:** $K_x$ is perfectly convex iff K is perfectly convex.

**Proof :** Assume that $K_x$ is perfectly convex. Let $(x_n)$ be a bounded sequence in K and let $\alpha_n$ be such that $\sum \alpha_n = 1$. Then $(x_n - x)$is a bounded sequence in $K_x$. Since $K_x$ is perfectly convex, we get that $\sum \alpha_n (x_n - x) \in K_x$

**Now,** $\sum \alpha_n = \sum \alpha_n (x_n - x + x)$

$= \sum \alpha_n (x_n - x) + \sum \alpha_n x$

$$= \sum \alpha_n (x_n - x) + x$$
$$\in K - x + x = K.$$

$\therefore K$ is perfectly convex.

Conversely let K be perfectly convex. Consider a bounded sequence $(y_n)$ kn $K_x$.

Let $\alpha_n \geq 0$ such that $\sum \alpha_n = 1$

$\therefore y_n = x_n - x$, where $x_n \in K$..

Since $(y_n)$ is bounded, we get that $(x_n)$ is a bounded sequence in K.

$(\|x_n\|) = \|y_n + x\| \leq \|y_n\| + \|x\| \leq M + \|x\| = M_1)$

Since K is perfectly convex, we get that $\sum \alpha_n x_n \in K$.

Theorem 9.1.3 If K is a perfectly convex set in a banach space X, then $\overset{0}{K} = \overset{C}{K} = \overline{\overset{c}{K}} = \overline{\overset{0}{K}}$

**Proof**: We will show that $(\overline{K})^0 \subseteq \overset{0}{K} \subseteq \overset{C}{K} \subseteq \overline{\overset{0}{K}} \subseteq (\overline{K})^0$. The middle two inclusions are trivial.

To show $\overset{0}{K} \subseteq \overset{C}{K}$,

Let $x \notin \overset{0}{K}$. Then we have an open ball $B(x, \in) \subseteq K$. Lety $\in X$.Choose

$$\alpha = \frac{\in}{2\|y - x\|}.$$

Then for $0 \leq \lambda \leq \alpha$, wehave $\|\lambda y + (1 - \lambda x) - x\| = \|\lambda (y - x)\|$

$$= \lambda \|y - x\|$$

$$\leq \frac{\in}{2}$$

$$\leq \in$$

$\therefore \lambda y+(1-\lambda)x \in B(x,\varepsilon)$, *for* $0 \leq \lambda \leq \alpha$

So $\lambda y+(1-\lambda)x \in K$, for $0 \leq \lambda \leq \alpha$

$\therefore$ we get that $x \in \overset{C}{K}$

Hence $\overset{0}{K} \subseteq \overset{C}{K}$.

Let $x \in \overset{c}{K}$, then $\forall y \in X, \exists \alpha > 0$ such that
$\lambda y+(1-\lambda)x \in K$, for $0 \leq \lambda \leq \alpha$

Since
$K \subseteq \overline{K}$, *we get that* $\forall y \in X, \exists \alpha > 0$ *such that* $\lambda y+(1-\lambda)x \in \overline{K}$, *for*
$0 \leq \lambda \leq \alpha$

$\therefore x \in \overline{\overset{C}{K}}$. Hence $\overset{C}{K} \subseteq \overline{\overset{C}{K}}$.)

For the first one it is enough to prove that $0 \in \left(\overline{K}\right)^0 \Rightarrow 0 \in \overset{0}{K}$.

For the last one it is enough to prove that $0 \in \left(\overline{\overset{C}{K}}\right) \Rightarrow 0 \in \left(\overset{C}{\overline{K}}\right)$.

Let
$0 \in \left(\overline{\overset{0}{K}}\right)$. Then 0 is an interior point of $\overline{K}$. Let $D_{\in}$ be an open ball of radius of $\in$
$> 0$ centered at 0 such that $D_{\in} \subseteq \overline{K}$.

This implies $D_{\in} \subseteq \overline{K \cap D_{\varepsilon}} \subseteq K \cap D_{\varepsilon} + \frac{1}{2} D_{\varepsilon}$

$\because n_0 \in N \text{ such that } \|x_{n_o} - x\| < \frac{\in}{2}$

Now, $x = x_{no} + (x - x_{no}) \in K \cap D_{\in} + \frac{1}{2} D_{\in}$

So, $\overline{K \cap D_{\in}} \subseteq K \bigcap D_{\in} + \frac{1}{2} D_{\in}$ ]

Then for every $\alpha > 0$,

$$\alpha D_{\in} \subseteq \alpha(K \cap D_{\in}) + \frac{\alpha}{2} D_{\in} \qquad (1)$$

Now, we show that $\frac{1}{2} D_{\in} \subseteq K$. So that 0 becomes an interior point of K, since $0 \in \frac{1}{2} D_{\varepsilon}$.

Let $y \in \frac{1}{2} D_{\in}$. Then we can write

$$y = \frac{1}{2} x_1 + y_1 \text{ where } x_1 \in K \cap D_{\in} \text{ and } y_1 \in \frac{1}{4} D_{\in.} \left[ \text{by taking } \alpha = \frac{1}{2} \text{ in (1)} \right]$$

Now, we repeat (1) for this $y_1$.

i.e, $y = \frac{1}{4} x_2 + y_2$ where $x_2 \in K \bigcap D_{\in}$ and $y_2 \in \frac{1}{8} D_{\in,}$ and so on

$\therefore y = \sum_{n=1}^{\infty} \left( \frac{1}{2^n} \right) x_n \in K, \text{since } (x_n) \text{ is bounded and the set K is perfectly convex.}$

Thus we proved the first inclusion.

Next, let $0 \in \overline{\overset{C}{K}}$. It follows from the definition of the kernel of K that

$$X=\bigcup_{n=1}^{\infty} n\left(\overline{K}\cap\left(-\overline{K}\right)\right) \qquad (2)$$

[Let x ∈ X.

Since $0\in\overset{C}{K},\exists\alpha>0$ such that $\lambda x+(1-\lambda)0\in\overline{K}$ for $0\le\lambda\le\alpha$

$\lambda(-x)\in\overline{K}$ *for* $0\le\lambda\le\alpha$.

*Since* $x\in X$ *we have* $-x\in X$.

i.e, $\therefore\exists\beta>0$ *such that* $\lambda x+(1-\lambda)0\in\overline{K}$ *for* $0\le\lambda\le\beta$.

*i.e.,* $-\lambda x\in\overline{K}$ *for* $0\le\lambda\le\beta$

Choose in

$\in$ N such that $\frac{1}{n}\le\min\{\alpha,\beta\}$. Then, we get $\frac{1}{n}x\in\overline{K}$ and $-\frac{1}{n}x\in\overline{K}$.

$$X\subseteq\bigcup_{n=1}^{\infty} n\left(\overline{K}\cap\left(-\overline{K}\right)\right).$$

I.e, x Thus, $X=\bigcup_{n=1}^{\infty} n\left(\overline{K}\cap\left(-\overline{K}\right)\right).$]

By the Baire Hausdorff theorem, X is of second category, meaning that one of the sets in the above union is not nowhere dense.

So for some n, the set $n\left(\overline{K}\cap\left(-\overline{K}\right)\right)$ is closed and not nowhere dense, and consequently it has an interior point.

This implies that the set $\overline{K}\cap\left(-\overline{K}\right)$ also has an interior point, say $x_0$,

i.e, for some $\in>0, D(x_0,\in)\subseteq\overline{K}$ and $-D_{(x_0,\in)}\subseteq\overline{K}$, where $D_{(x_0,\in)}$ is the $\in-$ball centered at $x_o$.

It follows that $D_{x_0(\in)} \subseteq \overline{K}$ and $-D_{(x_0,\in)} \subseteq \overline{K}$.

Since the set $\overline{K}$ is convex, we obtain,

$$D_{(0,\in)} \subseteq \frac{1}{2}D(x_0,\in) - \frac{1}{2}D(x_0,\varepsilon) \subseteq \frac{1}{2}\overline{K} + \frac{1}{2}\overline{K} \subseteq \overline{K}$$

$$\therefore 0 \in \left(\overline{K}\right)$$

$$Let\ x \in D(0,\in),$$

$$i.e. \|x\| < \in$$

$$\text{We have } x = \frac{1}{2}(x + x_0) - \frac{1}{2}(x - x_0) \in \frac{1}{2}D(x_0,\in) - \frac{1}{2}D(x_0,\in)$$

$$\therefore D(0,\varepsilon) \subseteq \frac{1}{2}D(x_0,\varepsilon) - \frac{1}{2}D(x_0,\varepsilon).$$

Let $x \in \frac{1}{2}\overline{K} + \frac{1}{2}\overline{K}.$

$$\therefore x = \frac{1}{2}y + \frac{1}{2}z \in \overline{K},\ where\ y, z \in \overline{K},\ Since\ \overline{K}\ is\ convex.]$$

$$\left[ Let\ x, y \in \overline{K}. Then\ \exists\ sequences\ (x_n), (y_n)\ in\ K\ such\ that\ x_n \to x\ and\ y_n \to y. \right.$$

Consider

$0 \le \lambda \le 1.$ *The sequence* $\left(\lambda x_n + (1-\lambda)y_n\right)$ *is in* $K$, since $K$ *is convex.*

*Also* $\lambda_{xn} + (1-\lambda)y_n \to \lambda x + (1-\lambda)y.$ $\therefore$, *we get* $\lambda x + (1-\lambda)y \in \overline{K}.$ $\overline{K}$ *is conv*

## 9.2 The Banach Open Mapping Theorem

**Theorem 9.2.1 (The banach Open Mapping Theorem)**

Let X, Y be the banach spaces. A : X $\rightarrow$ Y be a bounded linear operator onto Y.

Then A is an open map. i.e. if *O* is an open subset of X, then *A(O)* is an open subset of Y.

*Proof*: Let D be an open ball around *O*.

We note that *D* is perfectly convex.

$[$Let D be an open ball around O

We note that D is perfectly convex.

$[\text{Let } D = \{x; \|x\| < r\}.$

Let($x_i$) be a bounded sequence in *D* and ($\alpha_i$) be a real sequence such that $\alpha_i \geq 0 \, and \sum_{i=1}^{\infty} \alpha_i = 1.$ *Note that* $\|x_i\| < r, \forall_i$.

*case* 1; some $\alpha_k$=1. Then all the remaining $\alpha_{I's}$ are 0.

$$\therefore \sum_{i=1}^{\infty} \alpha_i x_i = \alpha_k x_k.$$

$$\therefore \left\| \sum_{i=1}^{\infty} \alpha_i x_i \right\| = \|\alpha_k x_k\| = \alpha_k \|x_k\| < 1.r = r$$

$$\text{so } \sum_{i=1}^{\infty} \alpha_i x_i \in D.$$

*Case* 2 ; $\alpha_i$<1, $\forall$i.

Then,

$$\|\alpha_1 x_1 + \alpha_2 x_2 + \ldots\ldots\ldots\ldots\| \leq \|\alpha_1 x_1\| + \|\alpha_2 x_2 + \alpha_3 x_3 + \ldots\ldots\|$$

$$< \alpha_1 .r + \left\| \alpha_2 x_2 + \alpha_3 x_3 + .. \right\|$$

$$\leq \alpha_1 .\mathrm{r} + \alpha_2 \left\| \mathrm{x}_2 \right\| + \alpha_3 \left\| \mathrm{x}_3 \right\| + .....$$

$$\leq \alpha_1 .\mathrm{r} + \alpha_2 \mathrm{r} + \alpha_3 \mathrm{r} + ..... = \mathrm{r}\left(\alpha_1 + \alpha_2 + \alpha_3 + .....\right) = \mathrm{r}$$

So, $\sum_{i=1}^{\infty} \alpha_i x_i \in \int^{x} D.]$

Since A is continuous, $F = A(D)$ is also perfectly convex.

[Let $(y_i)$ be a bounded sequence in A(D).

$\therefore (x_i)$ is a sequence in $D$ such that $Ax_i = y_i$.

Let $(\alpha_i)$ be a real sequence with $\alpha_i \geq 0$ and $\sum_{t=1}^{\infty} \alpha_i = 1$.

Now, $\sum_{i=1}^{\infty} \alpha_i y_i = \sum_{i=1}^{\infty} \alpha_i A x_i = \left( \sum_{i=1}^{\infty} \alpha_i x_i \right) \in A(D)$ since A is

*continuous and* $\sum_{i=1}^{\infty} \alpha_i x_i \in D,$ because D is perfectly convex.]

**Claim:** F coincides with its kernel.

**Proof :** Let $z \in F$. Then $\exists\ \omega \in D$ such that $A\omega = z$. We wish to prove that $z \in F^c$. Take any $y \in Y$. Since A is onto $\exists x \in X$. such that Ax=y. Since $\varpi \in D$ *and* $\overset{c}{D} = D, \exists\ \alpha > 0$ so that for all $0 \leq \lambda \leq \alpha$, we have

$\lambda x + (1-\lambda)\omega \in D.$

Applying A we get,

$$\lambda Ax + (1-\lambda) A\omega \in A(D)$$

i.e.,

$$\lambda y + (1-\lambda)z \in F,$$

$$\forall 0 \le \lambda \le \alpha$$

$\therefore z \in \overset{C}{F}$ So we have $F \subseteq \overset{C}{F}$. Clearly $\overset{C}{F} \subseteq F$.

$\therefore \overset{C}{F} = F$ and our claim is proved.

So by the theorem 9.1.3, we get $\overset{C}{F} = F$, i.e. F is open.

Now, assume that O is an open set in X.

Let $y \in A(O)$. i.e, $y = Ax$, $x \in D(x, \in) \subseteq O$, where $D(x, \in)$ is an open ball of radius $\in > 0$ centered at x.

Note that $A(D(x, \in)) - Ax = A(D(0, \in))$.

$\therefore Ax$ is an interior point of $A(D(x, \in))$ and hence y is an interior point of A(O)

$\therefore A(0)$ is open.

**Collary 9.2.2 (Bounded Inverse Theorem)**

Let X, Y be Banach space. Let $A \in L(X \mapsto Y)$ be one-one and onto. Then $\exists\ A^{-1} \in L(Y \mapsto X)$.

**Proof:** Since A is a bijective linear map from X to Y, $A^{-1} : Y \mapsto X$ is linear. Since A is open, we get that $A^{-1}$ is continuous.

**Remarks** : X, Y be Banach spaces. $A \in L(X \mapsto Y)$ be a bijection .Then A is a linear isomorphism between the spaces X and Y. In other words, A can be considered as the identity map between X and Y.

Since A is bounded, $\exists$ a constants C such that $\|x\|_Y \le C\|x\|_x$, for all $x \in X$. By bounded inverse theorem, we get that $\exists$ a constant $C_1$ such that $\|x\|_x \le C_1\|x\|_Y$, for all $x \in X$.

This means that A is also a topological isomorphism (i.e, a homeomorphism).

**Corollary 9.2.3** : Let X be a linear space equipped with two norms $\|.\|_1$ and $\|.\|_2$. Assume that $(X,\|.\|_1)$ and $(X,\|.\|_2)$ are complete normed spaces and that $\exists$ a constant C such that $\|x\|_2 \le C\|x\|_1$ for all $x \in X$. Then the norms $\|.\|_1$ and $\|.\|_2$ are equivalent.

**Proof :** Let us denote $(X,\|.\|_1)$ *as* $X$ *and* $(X,\|.\|_2)$ *as* $Y$.

Define the identy operator $I : X \mapsto Y$.

From the condition $\|x\|_2 \le C\|x\|_1$, for all $x \in X$; we get that I is continuous.

$\therefore$ by bounded inverse theorem $I^{-1} = I : Y \to X$ is continuous. So $\exists$ a constant $C_1$ such that $\|x\|_1 \le C_1\|x\|_2$, *for all* $x \in X$.

$\therefore \|.\|_1$ and $\|.\|_2$ are equivalent.

## 9.3 The closed grah Theorem

Let X, Y be a Banach spaces and $A : X \to Y$ be a linear operator. [We do not assume that A is bounded and we do not assume that the domain DomA, where its action is defined, is the whole space X. However as we consider linear opertors, we naturally assume that Dom A is a linear, not necessarily closed, subspace of X.]

The set $\Gamma(A) = \{(x, Ax); x \in DomA\} \subseteq X \times Y$ is called Graph of A.

We define the norm in X x Y by $\|(x,y)\| = \|x\| + \|y\|$

We say that A is a closed graph operator if $\Gamma(A)$ is a closed set in X x Y.

**Remark**: A is closed graph operator $\Leftrightarrow$ *whenever*

$x_n \in$ Dom A, $x_n \to x$ and $Ax_n \to y$ then $x \in$ Dom A and $Ax = y$

Proof: Assume that

$\Gamma(A) = \Gamma(A.)$. *Let* $x_n \in Dom\, A, x_n \to x$ *and* $Ax_n \to y$.

So, we have $\|(x_n, Ax_n) - (x, y)\| = \|(x_n - x, Ax_n - y)\|$

$= \|x_n - x\| + \|Ax_n - y\|$

$\to 0$ as $n \to \infty$

ie; $(x_{n,} Ax_n) \to (x, y)$ *in* $X \times Y$

Since $(x_n, Ax_n) \in \Gamma(A)$ and $\overline{\Gamma(A)} = \Gamma(A)$

We get $(x, y) \in \Gamma(A)$.

ie; $x \in$ Dom A and $Ax = y$

conversely assume that $x_n \in$ Dom A, $x_n \to x$, and

$Ax_n \to y$ imply $x \in$ Dom A and $Ax = y$

Let $(x,y) \in \Gamma(A)$. So $\exists$ a sequence $(x_{n,} Ax_n) \in \Gamma(A)$ s.t.

$(x_n, Ax_n) \to (x, y)$

ie; $\|(x_{n,} Ax_n) - (x, y)\| \to 0$ as $n \to \infty$.

Now, $\|x_n - x\| \le \|x_n - x\| + \|Ax_n - y\|$

$= \|(x_n, Ax_n) - (x, y)\|$

$\to 0$ as $n \to \infty$

Similarly $\|Ax_n - y\| \to 0$ as $n \to \infty$.

So we have $x_n \in$ Dom A, $x_n \to x$ and $Ax_n \to y$

Therefore we get $x \in$ Dom A and $y = Ax$.

So $(x, y) \in \Gamma(A)$.

Thus $\Gamma(A) \subseteq \Gamma(A)$ and hence $\Gamma(A)$ is closed.

**Note**: Consider a bounded operator $A \in L(X \mapsto Y)$ Then A is a closed graph operator

Proof : Let A be a bounded operator. Assume that

$x_n \in X$, $x_n \to x$ and $Ax_n \to y$. Since A is bounded and $x_n \to x$. We get $Ax_n \to Ax$.

Therefore we have Ax = y.

So, A is a closed graph operator.

Note: Assume that ker A = 0. Then $A^{-1}$ is formally defined on $ImA(DomA^{-1})$ and has a closed graph.

Proof : We have $\Gamma(A^{-1}) = \{(y, A^{-1}y); y \in ImA\}$

$= \{(Ax, x); x \in X\}$

Since $\Gamma(A) = \{(x, Ax); x \in X\}$ is closed in XxY, we get that $\Gamma(A^{-1})$ *is closed in* $Y x X$.

*Let* $y_n \in \text{Im}\, A, y_n \to y$ *and* $A^{-1}y_n \to x$ *Then we have* $x_n \in X$ *s.t.* $Ax_n = y_n$

So we get $x_n \in Dom\ A$, $x_n \to x$ *and* $Ax_n \to y$. Since A is closed, we get that

$Ax = y.ie, x = A^{-1}y$. Hence $A^{-1}$ is a closed graph operator.

**Definition 9.3.1**

We say that an operator A : X → Y admits a closure if and only if $\overline{\Gamma(A)} \subseteq$ X x Y is the graph of an operator.

**Remark**: A linear operator A admits a closure if and only if $Ax_n \to y$ and $x_n \to 0$ imply y = 0

Proof : Assume that A admits a closure. Let $x_n \to 0$ and $Ax_n$ y. since A admits a closure, ∃ an operator B such that $\Gamma(B) = \overline{\Gamma(A)}$. We have $(x_n, Ax_n) \in \Gamma(A)$, and $(x_n, Ax_n) \to (0, y)$

Therefore $(0, y) \in \overline{\Gamma(A)}$ ie, $(0, y) \in \Gamma(B) \Rightarrow B(0) = y$

So, we get $y = 0$.

Conversely assume that whenever $Ax_n \to y$ and $x_n \to 0$ we must have $y = 0$.

Define

*DomB* = $\{x_n \in X; \exists$ x sequence $(x_n)$ in X s.t. $x_n \to x$ in X and $Ax_n$ converges in Y$\}$

Define $B : DomB \to Y$ as $Bx = \lim_{n \to \infty} Ax_n$

Caim : B is well defined.

Proof : Let $x_n \to x$ and $x_n' \to x$ where both $Ax_n$ and $Ax_n'$ converge say to $y_1$ and $y_2$.

Therefore $A(x_n - x_n') \to x$

where both $Ax_n$ and $Ax_n'$ converge say to $y_1$ and $y_2$.

Therefore $A(x_n - x_n') \to y_1 - y_2$ *and* $x_n - x_n' \to 0$.

So by our assumption, $y_1 - y_2 = 0$.

Therefore $y_1 = y_2$. Thus our claim is proved. Clearly B is linear.

Claim : $\Gamma(B) = \Gamma(A)$

Proof : Let $(x, y) \in \overline{\Gamma(A)}$ $\therefore \exists$ a sequence

$(x_n, Ax_n) \in \Gamma(A)$ s.t. $(x_n, Ax_n) \to (x, y)$. so, $x_n \to x$ and $Ax_n \to y$ $\therefore$ by the definition of B, we get

$x \in$ DomB and $Bx = y$. ie; $(x, y) \in \Gamma(B)$. $\therefore$ we gt $\overline{\Gamma(A)} \subseteq \Gamma(B)$. Now let $(x, y) \in \Gamma(B)$. so we have $x \in$ DomB and $Bx = y$ $\therefore$ $\exists$ a sequence ($x_n$ in X such that $x_n \to x$ and $Ax_n$ converges in Y and $y = \lim_{n \to \infty} Ax_n$. We have $(x_n, Ax_n) \in \Gamma(A)$

and $(x_n, Ax_n) \to (x,y)$. This implies $(x,y) \in \Gamma(A)$. $\therefore$ we get $\Gamma(B) \subseteq \Gamma(A)$

Thus our claim is proved. Hence we get that a admits a closure.

Example : i)

The operator $Ax = \frac{dx}{dt}$ in $C[0,1]$ with

$DomA = C'[0,1] = \{x \in C[0,1] : x' \in C[0,1]\}$ *is a closed operator*.

Proof: From calculus, we know that if $x_n(t) \to x(t)$,

$X_n'(t) \to y(t)$ uniformly in [0,1] then x(t) is continuously differentiable and

x'(t) = y(t)

**Remark:** The above operator is not bounded.

Proof : Take $x_n(t) = t^n$, for $n = 1,2, \ldots$

Then $\|x_n\| = 1$, for all $n = 1,2, \ldots$

Now $Ax_n = nt^{n-1}$, for n = 1, 2, ….

n ||=n, for all an = 1, 2, . .

If possible, $\exists$ a constant c s.t. $\|Ax\| \le c\|x\|$ for all $x \in C'[0,1]$.

Then $\|Ax_n\| \le c\|x_n\|$ for all $n = 1,2,..$

Ie; n ≤ c for all n = 1.2, . . which is not true.

$\therefore$ A is not bounded.

ii) Consider the operator $Ax = \frac{dx}{dt}$ in $L_2[0,1]$ with

DomA = {x ∈ $L_2$[0,1]; x is absolutely continuous, x' ∈$L_2$[0,1], x(0) = 0}

The operator A has a bounded inverse $\left(A^{-1}y\right)(t)=\int_0^t y(s)\,ds.$

Hence A is a closed graph operator.

[Let A : X $\rightarrow$ Y be an operator such that

$A^{-1}$ : Y $\rightarrow$X is bounded. Then A is a closed graph operator.

Proof: Let $x_n \in$ DomA, $x_n \rightarrow x$ and $Ax_n \rightarrow y$. we wish to prove Ax = y.

Since $Ax_n \rightarrow y$ and $A^{-1}$ is bounded we get that $A^{-1}(Ax_n) \rightarrow A^{-1}(y)$.

ie; we have $x_n \rightarrow A^{-1}(y)$ . $\therefore$ we get $x = A^{-1}(y)$. so x $\in$ Dom A and Ax = y.]

iii) The operator $Ax=\frac{dx}{dt}$ in $L_2[0,1]$ with

Dom A = {x $\in$ C[0,1]; x' $\in$ C[0,1], x (0) = 0} is not closed, but admits a closure. Its closure is the operator defined in the previous example.

iv) Consider A : $L_2[0,1] \rightarrow L_2[0,1]$ with Ax(t) = x(0). t and DomA = C[0,1].

This operator des not admits a closure.

$$\text{proof}: x_n(t) = \left\{ \begin{array}{c} 1-nt \quad \text{if } 0 \le t \le \frac{1}{n} \\ 0 \text{ if } \frac{1}{n} \le t \le 1 \end{array} \right\}$$

Then we have $Ax_n(t) = t$, but $x_n(t) \rightarrow 0$ in $L_2[0,1]$

[we have $x_n(0) = 1$. $\therefore$ $Ax_n(t) = x_n(0).\ t = 1.t = t$. so $Ax_n \rightarrow$ The identity function on [0, 1]. But,

$$\|x_n\|^2 = \int_0^{1/n} (1-nt)^2 \, dt = \int_0^{1/n} (1-2nt+n^2t^2)dt$$

$$= \left[t - nt^2 + n^2t^3/3\right]_0^{1/n}$$

$$= \frac{1}{n} - n.\frac{1}{n^2} + n^2.\frac{1}{3n^3}$$

$$= \frac{1}{n} - \frac{1}{n} + \frac{1}{3n}$$

$$= \frac{1}{3n} \to 0 \text{ as } n \to \infty$$

**Theorem 9.3.2: (Closed Graph Theorem)**

Let, X, Y be Banach spaces. Let $A : X \to Y$ be a closed graph operator and

Dom A = X. Then A is a continuous operator.

Proof: Let $D_y$ be the open unit ball in Y centred at 0.

Claim: $A^{-1}(D_y)$ is perfectly convex.

Proof: Take any bounded sequence $\{x_n\}_{n=1}^{\infty} \subseteq A^{-1}(D_y)$.

Let $y_i = Ax_i \in D_y$. Let $\alpha_I \geq 0$ be such that $\sum\alpha_i = 1$. The series $\sum\alpha_i x_i$ is absolutely summable since $\{x_i\}$ is bounded.

$$\left[\sum\|\alpha_i x_i\| = \sum\alpha_i\|x_i\| \leq \sum\alpha_i k = k\sum\alpha_i = k < \infty\right]$$

Since X is a banach space, we get that $\sum\alpha_i x_i$ is convergent. Let $x = \sum\alpha_i x_i$

Similarly since $\{y_i\}$ is bounded and Y is a banach space, we get that $\sum\alpha_i y_i$ is convergent. Let $y = \sum\alpha_i x_i \in D_y$ since $D_y$ is perfectly convex. since A is closed graph operator, we get that $Ax = y$. $\therefore x \in A^{-1}(D_y)$

ie; $\sum \alpha_i x_i \in A^{-1}(D_y)$ and so that claim is proved. Now, since DomA = X we get that $0 \in (A^{-1})^C (D_y)$, the kernel of $A^{-1}(D_y)$ $\therefore$ by the theorem 9.1.3 we get $0 \in (A^{-1})^0 (D_y)$, the interior of $A^{-1}(D_y)$

ie; 0 is an interior point of $A^{-1}(D_y)$.

$\therefore$ A is continuous.

Remark : It is also possible to prove the closed graph theorem from the open mapping theorem with the following direct argument.

proof: Consider the subspace

E = {(x, y); y = Ax} $\hookrightarrow$ X × Y. If A has its graph closed, then E is a closed subspace of the complete space X ×Y and is therefore complete.

Define u : E $\rightarrow$ X as u(x, y) = x. The operator u is bounded, since

$\|u(x, y)\| = \|x\| \leq \|x\| + \|y\| = \|(x, y)\|$.

since, DomA = X, we get u is onto.

Now, $u(x_1, Ax_1) = u(x_2, Ax_2) \Rightarrow x_1 = x_2$.

$\therefore Ax_1 = Ax_2$. So $(x_1, Ax_1) = (x_2, Ax_2)$

$\therefore$u is one-one. Then we get $u^{-1}$ is bounded. Also the operator T : E $\rightarrow$ Y defined by T(x, y) = y is continuous. Now, $Tu^{-1}$ : X $\rightarrow$ Y is continuous.

Note that $Tu^{-1}(x) = T(x, Ax) = Ax$, for all $x \in X$.

$\therefore Tu^{-1} = A$ and thus A is continuous.

Theorem 9.3.4 : Let $X_0$, $X_1$, $X_2$ be Banach spaces and let $T_1$ : $X_0 \rightarrow X_1$, $T_2 : X_0 \rightarrow X_2$ be linear maps such that Dom $T_1 \subseteq$ Dom $T_2$, $T_1$ is a closed graph operator and $T_2$ admits a closure. Then there exists c > 0 such that

$\|T_2x\| \leq c(\|T_1x\| + \|x\|)$ for all $x \in \text{Dom}T_1$.

proof: Consider the closed subspace

$E = \{(x, T_1x); x \in \text{Dom } T_1\}$ of $X_0 \times X_1$.

set $V(x, T_1x) = T_2x$.

claim: V is a closed operator

proof: Assume that $(x_n, T_1x_n) \to (x, y)$ and

$V(x_n, T_1x_n) \to z$.

ie; $(x_n, T_1x_n) \to (x, y)$ and $T_2x_n \to z$.

we wish to prove that $V(x, y) = z$.

Since $(x_n, T_1x_n) \to (x, y)$, we have $x_n \to x$ and

$T_1x_n \to y$, since $T_1$ is closed, we get $x \in \text{Dom } T_1$ and $T_1x = y$.

since $T_2 x_n \to z$, we have $T_2 (x_n - x) \to z - T_2x$. Also

$x_n - x \to 0$. since $T_2$ admits a closure, we get that

$z - T_2x = 0$. $\therefore T_2x = z$. ie; $V(x, T_1x) = z$.

ie; $V(x, y) = z$.

so, our claim is proved.

moreover Dom V = E.

$\therefore$by closed graph theorem, V is continuous.

ie; $\exists$ a constant c such that

$\|V(x, T_1x)\| \leq c \|(x, T_1x)\|$

ie; $\|T_2x\| \leq c[\|x\| + \|T_1x\|]$

**EXAMPLE :**

For any $x \in C^{"}[0, 1]$, ie; for any twice continuously differentiable function on [0. 1], we have

$\| x' \| \leq M (\| x'' \| + \| x \|)$, where $\| . \|$ is the norm of the space C[0, 1].

proof: Let $X_0 = X_1 = X_2 = c[0, 1]$ $DomT_1 = C''+[0, 1]$,

$DomT_2 = C' [0, 1]$.

consider, $T_1 : X_0 \rightarrow X_1$ defined as $T_1x = x''$ and

$T_2 : X_0 \rightarrow X_2$ defined as $T_2x = x'$

we note that $DomT_1 = C''[0, 1] \subseteq C'[0, 1] = DomT_2$ ; $T_1$ is a closed graph operator and $T_2$ is a closed graph operator ($\therefore$obviously $T_2$ admits a closure.)

$\therefore$ by the above theorem $\exists$ a constant M such that

$\| x' \| \leq M (\| x'' \| + \| x \|)$.

**Theorem 9.3.5** : Let $E_1$, $E_2$ be closed subspaces of a Banach space X with $E_1 \cap E_2 = \{0\}$ and $E_1 + E_2 = X$. Then the projectin $P : X \rightarrow E_1$, parallel to $E_2$

(ie; $kerP = E_2$) is bouned operator. In other words, there exists $c > 0$ such that

$$c \| x_1 + x_2 \| \geq \max (\|x_1 \|, \| x_2 \| \text{ for } x_1 \in E_1 \text{ and } x_2 \in E_2.$$

proof: Assume that, $x^{(n)} = x_1^{(n)} + x_2^{(n)}$

where $x_i^{(n)} \in E_i : x^{(n)} \rightarrow x$ and $Px^{(n)} = x_1^{(n)} \rightarrow x_1$.

claim : $(x_2^{(n)})$ is convergent.

**Proof:**

$\| x_2^{(n)} - x_2^{(m)} \|$

$= \|x_2^{(n)} + x_1^{(n)} - x_2^{(m)} - x_1^{(m)} - x_1^{(n)}\|$

$= \|x^{(n)} - x^{(m)} + x_1^{(m)} - x_1^{(n)} \|$

$\leq \| x^{(n)} - x^{(m)} \| + \|x_1^{(m)} - x_1^{(n)} + \|$

$\rightarrow 0$ as $m, n \rightarrow \infty$

$\therefore (x_2^{(n)})$ is cauchy. Since X is complete, we get that $(x_2^{(n)})$ is convergent. So our claim is proved.

Assume that $x_2^{(n)} \to x_2$

claim : $x = x_1 + x_2$

**Proof:**

$$\| x^{(n)} - (x_1 + x_2) \| = \| x_1^{(n)} + x_2^{(n)} - x_1 - x_2 \|$$
$$\leq \| x_1^{(n)} - x_1 \| + \| x_2^{(n)} - x_2 \|$$
$$\to 0, \text{ as } n \to \infty$$

$\therefore (x_2^{(n)}) \to x_1 + x_2$

But we have already have $x^{(n)} \to x$.

$\therefore$we get $x = x_1 + x_2$ and the claim is proved. Since $E_1$ and $E_2$ are closed, we get that $x_1 \in E_1$ and $x_2 \in E_2$.

$\therefore Px = x_1$

Since DomP = X, we get, by the closed graph theorem, that P is bounded.

$\therefore \exists\ m > 0$ such that $\| Px \| = \| x_1 \| \leq m \| x \|$

$$= m(\| x_1 + x_2 \|)$$

Using the same arguments for the projection Q = I - P, parallel to $E_1$, we get that Q is a bounded operator.

$\therefore \exists\ m_1 > 0$ such that $\| Qx \| = \| x_2 \| \leq m_1 \| x \| = m_1 (\| x_1 + x_2 \|)$

Thus we have a constant $c > 0$ such that

$\| x_1 \| \leq c(\| x_1 + x_2 \|)$ and $\| x_2 \| \leq c(\| x_1 + x_2 \|)$.

$\therefore \max (\| x_1 \|, \| x_2 \|) \leq c(\| x_1 + x_2 \|)$.

**Definition 9.4.1** A non negative function $\mu$ on a Banach space X is called perfectly convex

if $\mu\left(\sum_{n=1}^{\infty} a_n xn\right) \le \sum_{n=1}^{\infty} a_n \mu(x_n)$ for all

$a_n \ge 0$ satisfying $\sum_{n=1}^{\infty} an = 1$ and for any bounded sequence $(x_n)$.

**Theorem 9.4.2** Let $\mu$ be a function on a Banach space, X, $\mu(\lambda x) = \lambda\mu(x)$ for any $\lambda \le 0$ and assume that $\mu$ is perfectly convex. Then there exist C > 0 such that $\mu(x) \le$ C || x || for all x ∈ X

**Proof:** Consider the set M = {x; $\mu$ *(x)* ≤ 1}.

We note that 0 ∈ $M^c$.

(Let y ∈ X.

Case(i) : $\mu$ *(y)* = 0. Then

$\mu(\lambda y + (1-\lambda)0) = \mu(\lambda y) = \lambda\mu(y) = 0 \le 1$

$\therefore \lambda y + (1-\lambda)0 \in M, \forall \lambda.$

So 0 ∈ $M^c$

Case (ii): $\mu(y) \ne 0$. Choose $\alpha = \dfrac{1}{\mu(y)}$

Then for

$0 \le \lambda \le \alpha$, we have $\mu(\lambda y + (1-\lambda)0) = \mu(\lambda y) = \lambda\mu(y) \le \alpha\mu(y) = 1$

$\therefore \lambda y + (1-\lambda)0 \in M, \forall 0 \le \lambda \le \alpha$

So 0 ∈ $M^c$.)

Claim : M is perfectly convex.

Pf: Let $(x_n)$ be a bounded sequence in M and let $\alpha_n \ge 0$ be such that $\sum_{n=1}^{\infty} \alpha_n = 1$. Since each $x_n$ ∈ M, we have $\mu(x_n) \le 1$.

Now,

$$\mu\left(\sum_{1}^{\infty}\alpha_n x_n\right) \le \sum_{n=1}^{\infty}\alpha_n \mu(x_n), \text{ since } \mu \text{ is perfectly convex}$$

$$\le \sum_{n=1}^{\infty}\alpha_n .1 = 1$$

$\therefore \sum_{1}^{\infty}\alpha_n x_n \in M$ and hence our claim is proved.

By theorem 9.1.3, $M^0 = M^c$ . $\therefore$ we have $0 \in M^0$

$\therefore \exists\, \delta > 0$ such that $D_\delta = D(0,\delta) \subseteq M.\left(D(0,\delta) = \{x : \|x\| \le \delta\}\right)$

In other words, $\exists \delta > 0$ such that $\|x\| < \delta$ implies $\mu(x) \le 1$.

Again, using the condition

$\mu(\lambda x) = \lambda\mu(x)$, we conclude that $\mu(x) \le \frac{1}{\delta}\|x\|$, for all $x \in X$.

(If x = 0, then

$\mu(x) = 0$. So $\mu(\lambda x) \le \frac{1}{\delta}\|x\|$. Let $x \ne 0$. Consider $\frac{x}{\delta\|x\|}$.

Note that $\left\|\frac{\delta x}{\|x\|}\right\| = \delta$

$$\therefore \mu\left(\frac{\delta x}{\|x\|}\right) \le 1 \to \mu(x) \le \frac{1}{\delta}\|x\|$$

**Theorem 9.4.3:** Banach-Steinhaus Theorem) Uniform Boundedness Theorm

Let $\{A_\alpha : X \mapsto Y\}$ $\alpha$ be a family of bounded operators between two Banach space X and Y, and assume that for

any $x \in X$ there $C_x > 0$, such that $\sup_\alpha \|A_\alpha x\| < C_x$.

Then there exists $C > 0$ such that $\|A_\alpha x\| \le C\|x\|$ for every x and $A_\alpha$ from the family. This means that the set of operators $\{A_\alpha\}$ is a bounded set in L (X ↦ Y).

(i.e; $\exists C > 0$ such that $\|A_\alpha\| \le C$, for every $\alpha$)

Proof: Introduce the function $\mu(x) = \sup_\alpha \|A_\alpha x\|$ (This function is defined properly. Since $\sup_\alpha \|A_\alpha x\|$ exists, for every x.)

$\mu$ satisfies all the conditions of theorem 9.4.2

(For $\lambda \ge 0, \|A_\alpha(\lambda x)\| = \lambda \|A_\alpha x\|. \therefore \sup_\alpha \|A_\alpha(\lambda x)\| = \lambda \sup_\alpha \|A_\alpha x\|$

i.e.; $\mu(\lambda x) = \lambda \mu(x)$ for $\lambda \ge 0$

Let $(x_n)$ be a bounded sequence in X, $a_n \ge 0$ be such that $\sum_{n=1}^{\infty} a_n = 1$

Then for any $\alpha$, we have

$$\left\|A_\alpha\left(\sum_{n=1}^{\infty} a_n x_n\right)\right\| = \left\|\sum_{n=1}^{\infty} a_n A_\alpha(x_n)\right\| \le \sum a_n \|A_\alpha(x_n)\| \le \sum a_n \sup_\alpha \|A_\alpha(x_n)\|$$

$$= \sum_{n=1}^{\infty} a_n \mu(x_n)$$

$$\therefore \sup_\alpha \left\|A_\alpha\left(\sum_{n=1}^{\infty} a_n x_n\right)\right\| \le \sum_{n=1}^{\infty} a_n \mu(x_n)$$

i.e; $\mu\left(\sum_{n=1}^{\infty} a_n x_n\right) \le \sum_{n=1}^{\infty} a_n \mu(x_n)$. So $\mu$ is perfectly convex.)

$\therefore$by theorem 9.4.2, $\sup_\alpha \|A_\alpha x\| \le C \|x\|$, for any $x \in X$.

Hence $\|A_\alpha x\| \le C \|x\|$, for every $x \in X$ and any $A_\alpha$ from the family.

**Corollary 9.4.4**: Let X be a Banach space over the field F. Let $A \subset X^*$ be a set of bounded linear functionals such that for every $x \in X$, the set $\{f(x); f \in A\}$ is bounded. i.e; for

every $x \in X$, there exists a constant $C_x$ such that $\sup_{f \in A} | f(x) \leq C_x$. Then A is a bounded set in $X_*$;

i.e; $\exists$ a constant C such that $\|f\| \leq C$ for every $f \in A$.

**Proof:** Consider $f \in A$ as a linear operator $f : X \to F$ and apply theorem 9.4.3.

**Corollary 9.4.5 :** Let $A \subseteq X$ be a set in X such that for every $f \in X^*$ there exists a constant $C(f)$ such that $\sup_{x \in A} | f(x)| \leq C(f)$. Then A is norm bounded. i.e; $\exists$ a constant C such that $\|x\| \leq C$, for all $x \in A$.

**Proof:** Use the Banach-Steinhaus theorem for the family of linear operators $\{x : X^* \to F\}_{x \in A}$.

(Let $x \in A$. Define $j_x : X^* \to F$ as $j_x(f) = f(x)$. Then $j_x$ is a linear functional and $\|j_x\| = \|x\|$.

Take $\{A_\alpha\}\alpha = \{jx ; x \in A\}$

Note that $\sup_{x \in A} | j_x(f) = \sup_{x \in A} | f(x)| \leq C(f)$, for any $f \in X$.

$\therefore$by 9.4.3, we get that $\exists$ a constant C such that $\|j_x\| \leq C$, $\forall x \in A$.

i.e; $\|x\| \leq C$, $\forall x \in A$).

Corollary 9.4.6 : Let $A \subseteq L(X \mapsto Y)$ be a set of bounded operators such that for every $x \in X$ and every $f \in Y^*$ we have that $|f(Tx)| \leq C(x, f)$ for all $T \in A$. Then, there exists C such that $\|T\| \leq C$ for all $T \in A$.

Proof : For a fixed $x \in X$, using corollory 9.4.5. We obtain $\|Tx\| \leq C(x)$, for any $T \in A$. Now, from the Banach-Steinhaus theorem we obtain $\|T\| \leq C$, for all $T \in A$. (Consider a fixed $x \in X$.

Let $B = \{Tx; T \in A\} \subseteq Y$.

For every $f \in Y^*$, $|f(Tx)| \leq C(x, f)$ for all $Tx \in B$

$\therefore \sup_{Tx \in B} |f(Tx)| \leq C(x, f)$ .

So by corollary 9.4.5, $\exists$ a constant C(x) such that

$||Tx|| \leq C(x)$, for all $T_x \in B$

i.e; $||Tx|| \leq C(x)$, for all $T \in A$

$\therefore$by theorem 9.4.3, we get a constant C such that $||T|| \leq C(x)$, for all $T \in A$

**Remarks**: Let X be a banach space. Let $(A_n)$ be a sequence of operators in L(X). If $(A_n)$ is Cauchy in the strong sense then $\exists$ an operator $A \in L(x)$ such that $A_n \rightarrow A$ strongly.

(Pf: Let $(A_n)$ be cauchy in strong sense. i.e; $(A_n x)$ is a cauchy sequence for each $x \in X$... for each $x \in X$, $(A_n x)$ convergesin X. Define $Ax = \lim_{n\rightarrow\infty} A_n x$. Consider $\{A_n\}_{n=1}^{\infty}$ of bouded operators on X. Since $(A_n x)$ is a cauchy for each $x \in X$, we have $(||A_n x||)$ is bounded for each $x \in X$.

i.e; for each $x \in X$, $\exists\ C_x > 0$ such that $\sup_n ||A_n x|| \leq C_x$.

$\therefore$by 9.4.3, $\exists\ C > 0$ such that $||A_n x|| \leq C\,||x||$, $\forall x \in X$ and $\forall n$.

So $\lim ||A_n|| \leq C\,||x||$, $\forall x \in X$. i.e; $||Ax|| \leq C\,||x||$. So A is a bounded operator).

**Lemma 9.4.7 : A sequence of operators $T_n \in L(X, y)$ converges strong**ly to an operator $T \in L(X, Y)$ if and only if

1. the sequence $(T_n x)$ converges for any x from a dense subset of X;

2. there exist $c > 0$ such that $||T_n|| \leq C$.

**Proof:** If $(T_n x)$ converages for all $x \in X$, then (i) is clear and (ii) follows from the Banach steinbaus theorem.

Conversely, assume that $(T_nx)$ converges for any $x \in M$, where $\overline{M} = X$, and $||T_n|| \leq C$.

For any $x \in M$, define the operator $T_0$ via $T_0x = \lim_{n\to\infty} T_nx$. It is obvious that $T_0$ is linear.

Since $||T_n|| \leq C$, $\forall n$ we have $||T_nx|| \leq C\,||x||$, $\forall x$ and $\forall n$.

$\therefore \lim ||T_nx|| \leq C\,||x||$, $\forall x \in M$

i.e; $\lim ||T_nx|| \leq C\,||x||$, $\forall x \in M$ i.e; $||T_0x|| \leq C\,||x||$, $\forall x \in M$

So, we get that $||T_0|| \leq C$.

We extend the operator $T_0$ on the whole space X as follows:

For any $x \in X$, take $x_m \in M$ such that $x_m \to x$ and set $Tx = \lim_{m\to\infty} T_0x_m$. It is easy to check that the limit exists. does not depend on the choice of the sequence $\alpha_m$ and $||T|| \leq C$.

*($(T_0x_m)$*is a cauchy sequence in *X* since

$$\|T_0x_m - T_0x_k\| = \|T_0(x_m - x_k)\| \leq \|T_0\|\ \|x_m - x_k\| \leq C\|x_m - x_k\| \to 0.$$

So $\lim_{m\to\infty} T_0x_m$ exists.

Let $x_m \to x$ and $x'_m \to x$ as well. Then $x_m - x'_m \to 0$.

Now

$$\|T_0x'_m - Tx\| \leq \|T_0x'_m - T_0x_m + T_0x'_m - Tx\| \leq \|T_0\|\,\|x'_m - x_m\| + \|T_0x_m - Tx\| \to 0.$$

$\therefore \lim_{m\to\infty} T_0x'_m = Tx.$

Now,

$$\|Tx\| = \|\lim_{m\to\infty} T_0x_m\| = \lim_{m\to\infty}\ \|T_0x_m\| \leq lim_{m\to\infty}\ \|T_0\|\ \|x_m\| \leq$$

$$C\lim_{m\to\infty}\|x_m\| = C\|x\|).$$

Claim: For any $x \in X, T_nx \to Tx$.

Pf: For

$\in > 0$, take $x_0 \in M$ such that $\|x - x_0\| < \frac{\in}{4)(C+1)}$ and next take $N_0 > 0$ such that for any $n > N_0$,

$\|T_n x_0 - Tx_0\| < \frac{\in}{2}$. Thus, for any $n > N_0$, we obtain $\|T_n x_0 - Tx\| \le \|T_n x_0 - Tx_0\| +$

$\|T_n x_0 - Tx_0\| +$

$\|Tx_0 - Tx \le\| \ \|T_n\| \ \|x_n - x_0\| + \|T_n\| \ \|x - x_0\| + \|T_n x_0 - Tx_0\| + \|T\| \ \|x_0 - x\|$

$< C.\frac{\in}{4(C+1)} + \frac{\in}{2} + C.\frac{\in}{4(C+1)} < \frac{\in}{4} + \frac{\in}{2} + \frac{\in}{4} = \in$.

**Theorem 9.4.8:** Let $H$ be a Hilbert space and let $A: H \to H$ be a linear operator with $Dom\ A = H$ and $A = A^*$.i.e; $\langle Ax, y\rangle = \langle x, Ay\rangle$, for all $x, y \in Dom\ A = H$. Then $A$ is continuous.

**Proof:** We have $|\langle Ax, y\rangle| \le \|x\|\ Ay\ \| \le \|x\|.\ \|Ay\| \le \|Ay\|$, for all $x \in H$ with $\|x\| \le 1$ and for all $y \in H$. By corollary 9.4.5, we conclude that the set $\{Ax; \|x\| \le 1\}$ is bounded which implies $\|Ax\| \le C\ \|x\|$, for all $x \in H$. (Let $B = \{Ax : \|x\| \le 1\} \subseteq H$. By RRT every $f \in H^*$, can be associate with $y \in H$. we have $|f(x)| = |\langle Ax, y\rangle| \le |Ay| = C(f)$, for all $Ax \in B$. $\therefore \sup_{Ax \in B} |f(Ax)| \le C(f)$. $\therefore$ by 9.4.5, we get that $B$ is bounded. i.e., $\|Ax\| \le C$ for all $x$ with $\|x\| \le 1$. Thus, we get that $\|Ax\| \le C\|x\|$, for all $x \in H$.)

Bases in Banach Spaces Let $X$ be a Banach space.

Let $e = \{e_k\}1^{\infty}$ be a linearly independent complete system in $X$. Consider the subspace $M = Span\ \{e_1, e_2, ....\}$.

For any $x \in M$, we have $x = \sum_{k=1}^{n} a_k e_k$ with $a_k = e_k^*$ are linear functionals defined as $e_k^*(x) = a_k$.

Since $e$ is a linearly independent system. the functional $e_k^* \in X^*$ such that $e_k^*(e_n) = \delta_{kn}. e_k^*, k = \delta_{kn} = \begin{matrix} 1, \text{ if } k=1 \\ 0, \text{ if } k \neq n \end{matrix}$

However, these functionals are defined only on $M$ and are not necessarily bounded. The system $e$ is called a minimal system if there exist linear functionals

$$e_k^* \in X^* \text{ such that } e_k^*(e_n) = \delta_{kn.} = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}$$

1, 2, .... are called biorthogonal functionals.

On the subspace M, consider the linear projections $\bigcup_n \left( \sum_{k=1}^{m} a_k e_k \right) = \sum_{k=1}^{n} a_k e_{k,}$ for all $a_k \in \mathbf{F}$.

(If $m < n$, the values $a_k$ for k>m are assumed to be zero.

$e$ is a minimal system iff the operators $U_n$ are bounded.

Let $e$ be a minimal system.

Then for any $x \in M$, we have $x = \sum_{k=1}^{m} a_k e_K = \sum_{K=1}^{m} e_k^*(x) e_k$ and

$$\bigcup_n x = \sum_{k=1}^{m} e_k^*(x) e_{k.}$$

$$\therefore \left\| \bigcup_n X \right\| \leq \sum_{K=1}^{n} \|e_K\| \left\| e_k^* \right\| \|x\| = \left[ \sum_{K=1}^{n} \|e_K\| \left\| e_k^* \right\| \right] \|x\| = C_n \|X\|$$

Thus, we get that $U_n$ are bounded and $\|U_n\| \le C_n < \infty$.

Conversely assume that $U_n$ are bounded operators. We need to prove that all functionals $e^k$ are bounded. We have $U_n x - U_n - 1x = e^{n(x)} e_n$

Therefore,

$\|e_n e^* n(x)\| = \|(U_n - U_{n-1})\text{x}\| \le \|U_n - U_{n-1}\| \|\text{x}\|$

i.x, $|e^* n(x^1)\ \|en\|\ \ \le\|\ \ \|U_n - U_{n-1}\|\ \ \|x\|$

i.e, $|e^* n(x)| \le \dfrac{1}{\|en\|} \|\text{U}_\text{n} - U_{n-1}\| \|x\|$

So $e^n$ is a bounded linear functional on M. Since $e$ is a complete system, M is dense in $X$. Therefore we can extend the functional on the all space $X$, and we obtain $e^n \in \text{X}^*$

**Definition 9.5.2**

Let $X$ be a Banach space and $e = \{e_1, e_2, ....\}$ be a linearly independent complete system in $X$. We call $e$ *a* "basis" of $X$ if and only if for every $x \in X$, there exists a unique sequence $(a_n)$ in F such that $x = \sum_1^\infty a_k e_k$.

We call $e$ a Schauder basis if, in addition, $e$ is a minimal system.

**Theorem 9. 5.4.** Let $e$ be a complete linearly independent system in a Banach space $X$. Then, $e$ is a basis of X if and only if the projections $U_n$ defined above on the dense subset M=span $e$ are uniformly bounded.

i.e; there exists a constants C > 0 such that $||U_n|| \le c$, for all $n$.

**Theorem 9. 5.3.** Every basis of a Banach space is a Schauder basis.

**Proof:** Let *e* be a basis of a banach space X. Then by then above theorem $\{u_n\}_n^{\infty} = 1$ is uniformly bounded. Therefore $u_n^{'}s$ are bounded therefore by lemma 9.5.1 *e* is a minimal system. Hence e is a Schauder basis.

## 9.6 Linear Functional; The Hahn Banach Theorem

Let L be a linear space over *R*.

Definition : A real function $p(x) \geq 0$ is called sublinear if

(i) $p(x+p) \leq p(x) = p(y)$

(ii) $p(\lambda x) = \lambda p(x)\ \forall \lambda \geq 0$

**Note:** We recall that $L^{\#}$ denote the space of all linear functionals on L. Let $L_0$ be a subspace of L. We use the notation

$f / L_0 = f_0$ if $f_0 \in L^{\neq} 0$ and $f(x) = f 0(x)$ for all $x \notin L_0$. The functional f is called an extension of $f_0$ and $f_0$ is called the restriction of f on $L_0$

**Theorem 9.6.2 (Hahn-Banach Theorem):** Real case

Let $p(x) < \infty$ for all $x \in L$ be a sublinear function, let $L_0$ be a subspace of L and let $f_0 \in L_0^{\neq}$ be a linear functional defined on $L_0$. Assume that $f_0(x) \leq p(x)$ for all $x \in L_0$. Then there exists $f \in L^{\neq}$ $f \in L^{\#}$ such that

(i) $f(x) \leq p(x)$ for all $x \in L$; and (ii) $f / L_0 = f_0$.

**Definition 9.6 .3** Let L be a linear space over *C*.

A real function $p(x) \geq 0$ is called convex if

(i) $p(x+y) \le p(x) + p(y)$ for all $x, y \in L$; and

(ii) $p(\lambda x) = |\lambda| p(x)$, for all $x \in L, \lambda \in C$.

**Theorem 9.6.4: (Hahn-Banach Theorem)**

Let L be a linear space over C. Let *p(x)* be a convex function and $p(x) < \infty \forall x \in L$. Let $f_0$ be a linear functional defined on a subspace $L_0 \hookrightarrow L$ such that $|f_0(x)| \le p(x) \forall x \in L_0$. Then there exist a linear functional $f \in L^*$ such that

(i) $|f(x)| \le p(x)$ for all $x \in L_0$.

(ii) $f / L_0 = f_0$.

**Proof:** Let $\Phi_0(x) = \text{Re } f_0(x)$ which is real valued linear functional on $L_0$ considered as a linear space over R. Then, for every $x \in L_0$ we have $\Phi_0(x) \le |f_0(x)| \le p(x)$.

So by a real case of the Hahn Banach theorem, there is an extension $\Phi \in L^*$ of the functional $\Phi_0$ considered L as a linear space over R, and $\Phi(x) \le p(x)$.

(Notice now the connection between the complex linear functional $f(x) = \Phi(x) + i\Psi(x)$,

(where $\Phi = \text{Re } f, \Psi = \text{Im } f$) and its real part $\Phi(x)$.

$if(x) = f(ix) = \Phi(ix) + i\psi(ix)$.

$so \quad f(x) = \psi(x) - \Phi(x)$

thus $\text{Im } f(x) = -\Phi(ix))$

set $f(x) = \Phi(x) - i\Phi(x)$

Them f is a complex valued linear functional over C.

(first we note that, $f(ix) = \Phi(ix) - i\Phi(-x)$

$f(ix) = \Phi(ix) + i\Phi(x) - i\Phi(x)) = if(x).$

Now for every $x \in L$,

$f(x+y) = \Phi(x+y) - i\Phi\big(i(x+y)\big)$

$= \Phi(x+y) - i\Phi(ix+iy)$

$= \Phi(x) + \Phi(y) - i\Phi(ix) - i\Phi\ (iy)$

$= f(x) + f(y).$

Now let $\alpha \in R$ and $x \notin X$. Then we have,

$f(\alpha x) = \Phi(\alpha x) - i\left(\Phi(\alpha ix)\right.$

$= \alpha\Phi(x) - i\alpha\Phi(ix)$

$= \alpha\big(\Phi(x) - i\Phi(x)\big)$

$= \alpha\ f(x).$

Now let $\lambda = \alpha + i\beta$, where $\alpha, \beta \in R$, and $x \in X$.

then $f(\lambda x) = f\big((\alpha + i\beta)x\big)$

$= f(\alpha x + i\beta x)$

$= f(\alpha x) + f(\beta ix)$

$= \alpha\ f(x) + \beta\ f(ix)$

$= \alpha\ f(x) + i\beta\ f(x)\ = (\alpha+i\beta).\ f(x)$

$= \lambda f(x))$

f is an extension of $f_0$, since for any $x \in L_0$ we have,

$f(x) = \text{Re}\ f(x) + i\,\text{Im}\ f(x) = \Phi(x) + i\Phi(ix) = \Phi_0(x) - i\Phi_0(ix) =$

Re $f_0(x) + i \operatorname{Im} f_0(x) = f_0(x)$.

Finally we check that $|f(x)| \le p(x)$, for every $x \in L$

we write, $f(x) = |f(x)| e^{i\theta(x)}$.

i.e, $f\left(e^{-i\theta(x)} x\right) = |f(x)|$

thus, $|f(x)| = f\left(e^{-i\theta(x)} x\right)$

$= \Phi\left(e^{-i\theta(x)} x\right)$

$\le p\Phi\left(e^{-i\theta(x)} x\right)$

$= \left|e^{-i\theta(x)}(x)\right| p(x)$

$= p(x)$

($f(e^{-i\theta(x)} x$ is real valued. So its imaginary part will be 0.

**Corollary 9.6.5:** Let X be a normed space let $E_0 \hookrightarrow x$ be a subspace and let $f \in x$ such that $f / E_0 = f_0$ and $\|f\| = \|f_0\|$.

**Proof:** Use the Hahn Banach theorem for $p(x) = \|f_0\| \|x\|$

**Corollary 9.6.6** :For every $x_0$ with

$\|x_0\| =$ I, there exists $f \in X^*, \|f\| = 1, f(x_0) = 1$.

(So, $f$ is a support functional at $x_0 \in s(x)$)

**Proof:** Consider the 1-demensional subspace $E_0 = \{\lambda x_0 ; \lambda \in C\}$ and the functional $f_0(x) = \lambda$, $f$ or $x = \lambda x_0$.

Clearly $f_0 \in E_0^*$ and $\|f\| = 1$. Also note that $f_0(x_0) = 1$ so, by the above corollary there exist $f \in X^*$ with the required properties.

**Corollary 9.6.7** : For every $x_0$ with there exist $f \neq 0$ such that $f(x_0) = \|x_0\| \, \|f\|$.

**Proof:** For $x_0$ =0. the result is trivial. So let $x_0 \neq 0$. Consider $\frac{x_0}{\|x_0\|}$ which has norm 1. Then apply the above corollary to get an $f \in X^*$ with $\|f\| = 1$ and $f\left(\frac{x_0}{\|x_0\|}\right) = 1$. There for we get $f(x_0) = \|x_0\| \|f\|$.

**Corollary 9.6.8** For every $x_1 \neq x_2$, there exists $f \in X$ such that $f(x_1) \neq f(x_2)$. This means that *X* is a total set.

**Proof:** Consider $x_1 - x_2 \neq 0$. By the above corollary there exist a nonzero $f \in X^*$ such that $f(x_1 - x_2) = \|x_1 - x_2\| \|f\|$. There for we get $f(x_1 - x_2) \neq 0$. i.e; $f(x_1) \neq f(x_2)$.

**Corollary 9.6.9:** If $X^*$ is a separable space, then X is also separable.

**Proof:** Let $f_1, f_2, \ldots$ be a dence set in S (X), the unit sphere of X. Choose $x_i \in S(x)$ such that $|f_i(x_i)| \geq \frac{1}{2}$, for $i = 1,2,\ldots\{$since $X^*$ is separable, every subset of *X* is separable. Therefore, *S (X)* is separable and has a countable dence subset if

$|f_i(x)| < \frac{1}{2}$ for all $x \in S(X)$, then $\|f_i\| \leq \frac{1}{2}$, which is not true. Therefore $\exists x_i \in S(X)$ such that $|f_i(x_i)| \geq \frac{1}{2}\}$

Let $E = span\{x_1, x_2, \ldots\}$

Claim: E=X.

**Proof:** If possible, assume that $E \neq X$. Then by corollary 3.1.7, there exists $f \in X^*$ such that $\|f\| = 1$ *and* $f(E) = 0$ . (since $E \neq X$ there exist $x \in X$ with $dist\,(x, E) = d > 0$. Therefore, corollary 3.1.7 can be applied). now for any $0 < \varepsilon < \frac{1}{2}, \exists f_{i_0}$ such that $\varepsilon \geq \|f - f_{i_0}\| \geq \left| f - f_{i_0}(x_{i_0}) = \left| f_{i_0}(x_{i_0}) \right| \geq \frac{1}{2}, \right|$ which is a contradiction. Therefore our claim is proved.

Now let $E_1$ be the set of all finite linear combinations of $x_1, x_2, \ldots$ with rational coefficients. Then clearly $E_1$ is countable and $\overline{E_1} = E$. Since $\overline{E} = X$, weget $\overline{E_1} = X$. Hence we get that X is separable.

**Corollary 9.6.10:** Let $E \hookrightarrow X$ be a closed subspace. Then,

(i) $(X/E^*) = E^{\perp} \hookrightarrow X^*$, and

(ii) $E = X' E^{\perp}$

(The equalities hear means that ∃ linear isometric isomorphisms between the corresponding normed spaces)

**Proof**: (i) for a given $f \in (X/E)^*$, define $f^{\sim} \in E$ via the equality $f^{\sim}(x) = f([x])$, for $x \in$. The map $f \mapsto f^{\sim}$ is linear one-one and onto E. ($\Phi : (X/E)^* \to (X)^*$ is defined as $\Phi(f) = f^{\sim}$

let $f_1, f_2 \in (X/E)^*$ then for any $x \in X$,

$(f_1 + f_2)^{\sim}(x) = (f_1 + f_2)([x]) = f_1([x]) + f_2([x]) = f_1^{\sim}(x) + f_2^{\sim})(x)$ therefore, $(f_1 + f_2)^{\sim} = f_1^{\sim} + f_2^{\sim}$

so $\Phi(f_1 + f_2) = \Phi(f_1) + \Phi(f_2)$ similarly, $\Phi(\alpha f) = \alpha\Phi(f)$. so $\Phi$ is linear. Let $\Phi(f_1) = \Phi(f_2)$. ie. $f_1^{\sim} = f_2^{\sim}$

Therefore, $f_1^{\sim}([x]) = f_2^{\sim}([x])$ for all $x \in X$

ie; $f_1([x]) = f_2([x])$ for all $[x] \in X/E$. Therefore $f_1 = f_2$ so $\Phi$

is one-one. Next we show that Im = $E^{\perp}$. Let $f^{\sim} \in$ im. So $\exists f \in (X/E)^{*}$ such that $\Phi(f) = f^{\sim}$. For $x \in E$, we have $f^{\sim}(x) = f([x]) = f([0]) = 0$. therefore $f^{\sim} \in E^{\perp}$. So Im $\subseteq E^{\perp}$. Conversely, let $f^{\sim} \in E^{1}$ define $f \in (X/E)^{*}$ as $f([x]) = f^{\sim}(x)$. Then clearly $\Phi(f) = f^{\sim}$. So $f^{\sim} \in Im\ \Phi$. Therefore $E^{\perp} \subseteq im$. Thus $Im\ \Phi = E^{\perp}$ and hence $\Phi$ is onto)

Finally, we show that $\|f\| = \|f^{\sim}\|$ (so that $\Phi$ becomes an isometry)

for any $z \in E$. We have $|f([x])| = |f^{\sim}(x)| = |f^{\sim}(x + x)| \leq \|f^{\sim}\|(x + z\|$

taking the infinum over E, we get,

$|f([x])| \leq \|f^{\sim}\|[x]\|$

therefore we get $\|f\| \leq \|f^{\sim}\|$ for the opposite direction we have,

$|f^{\sim}(x)| = |f([x])| \leq \|f\|\ \|[x]\| \leq \|f\|\ \|x\|$ so, $\|f^{\sim}\| \leq \|f\|$ thus, we get $\|f^{\sim}\| = \|f\|$

(ii) For $g^{\sim} \in E^{*}$, by the Hanh-Banh-Banach theorem $\exists g \in E^{*}$ such that $g / E = g^{\sim}$ and $\|g^{\sim}\|$

Consider the map $g^{\sim} \mapsto [g] \in X^{*} / E^{\perp}$. The map is well defined.

The map one-one because $[g_1] = [g_2]$ $g_1 = g_2 \in E^{1}$

$(g_1 - g_2)(x) = 0$, for all $x \in E$

$g_1(x) = g_2(x)$, for all $x \in E$

$g_1/E = g_2/E$

$g^{\sim}_1 = g^{\sim}_2$

The map is clearly onto because for any $[g] \in X^{*}/ E^{\perp}$, we can set $g^{\sim} = g/E$ and we get that the image $E^{*}[g]$. The linearity of the map is clear.

(let $\Psi : E^{*} \rightarrow X^{*}/E^{\perp}$ be the map given by $\Psi(g^{\sim}) = [g]$

For $\tilde{g} \in E^*$, let $g_1 \in E^*$ be such that $g_1/E = \tilde{g}$ and

For $\tilde{g}_2 \in E^*$, let $g_2 \in E^*$ be such that $g_2/E = \tilde{g}$

Then for $\tilde{g}_1 + \tilde{g}_2$, clearly $g_1 + g_2 \in E^*$ is such that $g_1 + g_2/E - \tilde{g}_1 + \tilde{g}_2$.

Therefore, we get

$\Psi(\tilde{g}_1 + \tilde{g}_2) = [g_1 + g_2] = [g_1] + [g_2] = \Psi(\tilde{g}_1) + \Psi(\tilde{g}_2)$

Similarly, $\Psi(\alpha\tilde{g}) = \alpha\Psi(\tilde{g})$

Finally, we prove that,

$\tilde{g} = [g]$

we have $[g]\, g = \tilde{g}$ ; and for every $g_0 \in E^1$,

$$\tilde{g} = \sup_{x \in E, \|x\|=1} \left| \tilde{g}(x) \right|$$

$$= \sup_{x \in E, \|x\|=1} \left| (g + g_0)(x) \right|$$

$$= \sup_{x \in E, \|x\|=1} \left| (g + g_0)(x) \right|$$

$= \|g + g_0\|$

Taking the infimum over all $g_0 \in E^{\perp}$, we get $\|\tilde{g}\| \leq \|[g]\|$. Thus we get that $\|\tilde{g}\| = \|[g]\|$

**corollary 9.6.11:** Let X be a reflexive space. Then, every closed subspace $E \hookrightarrow X$ is also reflexive.

**Proof:** From corollary 9.6.10, we have $E^* = X^*/E^{\perp}$ and

$$(X^*/E^{\perp})^* = (E^I)^{1} = (E^I)^{\perp}, \text{ since X is reflexive}$$

=E, by corollary 3.1.8.

## 10.1. BANACH ALGEBRAS

**Definition**: A Banach space A is called a Banach algebra if an operation "product" x. y is defined for the elements of a satisfying the following properties.

(i) The product is linear with respect to both variables.

i.e, $x.(y+z) = x.y + x.z$ and

$(x+y).z = x.z + y.z$

$(\alpha x).y = \alpha(x.y) = x.(\alpha y), \forall x, y, z \in A, \alpha \in k.$

(ii) The product is associative.

i.e, $(x.y).z = x.(y.z) \forall x, y, z \in A$

(iii) The product is continuous with respect to every variable (x and y).

(iv) The product has an identity element $e \in A$

i.e, $x.e = e.e.x = x, \forall x \in A$

**Remark**: The Banach algebra A is commutative if $x.y = y.x$ $\forall x, y, \in A$

**Theorem**: Let A be a Banach algebra. Then there exists an equivalent norm |x| on A such that

$|x.y| \leq |x|.|y|$ and $|e| = 1$.

**Proof:** For $x \in A$, define a linear operator $T_x : A \to A$ as $T_x(y) = x.y, y \in A$

Since x. y is continuous with respect to y, the linear operator $T_x$ is continuous and hence bounded.

(Let $y_n \to y$ in A. Since x.y is continuous with respect to y, we get that $x.y_n \to x.y$ in A. i.e, $T_x(y_n) \to T_x(y)$ in A. Therefore $T_x$ is continuous.).

Now, we define $\phi : A \to L(A)$ as $\phi(x.y) = \phi(x)\phi(y)$

[$\forall\ x_1, x_2 \in A$ and $\forall\ y \in A$

$$\phi\ (x_1 + x_2)\ (y) = Tx_1 + x_2\ (y) = (x_1 + x_2).\ y = x_1.\ y + x_{1\ -\ y.}$$

$$= Tx_1\ (y) + Tx_2\ (y)$$

$$= \phi\ (x_1)\ (y) + \phi\ (x_2)\ (y)$$

$$= \phi\ (x_1) + \phi\ (x_2))\ (y)$$

$\therefore \phi\ (x_1 + x_2) = \phi\ (x_1) + \phi\ (x_2) \quad \forall\ x_1, x_2 \in A$

Now, $\forall\ x \in A$, $\alpha \in k$ and $y \in A$

$\phi(\alpha x)(y) = T_{\alpha x}(y) = (\alpha x)\ .\ y = \alpha\ (x\ .\ y) = \alpha T_x\ (y)$

$= \alpha\phi(x)(y)$

$\therefore \phi(\alpha x) = \alpha\phi(x), \quad \forall\ \alpha \in k$ and $\forall\ x \in A$

So $\phi$ is linear.

$Tx.y(y) = (x.y)\ .\ z = x.\ (y.z) = x.\ \ T_y(z) = T_xT_y(z), \forall\ x,y, z, \in A$

$\therefore T_{x.y} = T_xT_y$

So we get $\phi(x.\ y) = \phi(x)\phi(y)$]

Let $P = \{B \in L\ (A);\ B\ (x\ .\ y) = B(x)\ .\ y\}$

Clain : $P = \phi(A)$.

Proof: Let $\phi\ (x) \in \phi\ (A)$. i.e, $T_x \in \phi\ (A)$

Then $T_x(y\ .\ z) = x\ .\ (y\ .\ z) = (x\ .\ y).\ z = T_x(y).\ z$

Now, let $B \in P$. Let $x = B\ (e)$. Then we get that

$$B(z) = B(e\ .\ z) = B(e).\ z = x\ .\ z = T_x(z)$$

$\therefore$ we have $B = T_{x.}$ So $B \in \phi\ (A)$

Hence the claim is proved.

Next, we prove that P is a closed subspace of L(A). Let $(B_n)$ be a sequence in P and $B_n \rightarrow B$ in the operator norm.

We have $B_n(x \,.\, y) = B_n(x).\, y$ for any $x, y \in A$. We have $B_n(x) \to B(x)$ and $B_n(x \,.\, y) \to B(x \,.\, y)$.

Therefore we obtain $B(x \,.\, y) = B(x) \,.\, y$

So $B \in P$. Thus P is a closed subspace of L(A).

Now, we define a new norm in A as $\|x\|_{op} = \|T_x\|$.

[Clearly $\|x\|_{op} \geq 0$. If $x = 0$, then $T_x$ = zero operator, and so $\|x\|_{op} = 0$.

Let $\|x\|_{op} = 0$. Then $T_x = 0$. i.e, $T_x(y) = 0 \;\forall\; y \in A$

Implies in particular $T_x(e) = 0$. i.e, $x \,.\, e = 0 \Rightarrow x = 0$.

$\|x + y\|_{op} = \|T_{x+y}\| = \|T_x + T_y\| \geq \|T_x\| + \|T_y\| = \|x\|_{op} + \|y\|_{op}$

$\|\alpha x\|_{op} = \|T_{\alpha x}\| = \|\alpha T_x\| = |\alpha| \, \|T_x\| = |\alpha| \, \|x\|_{op}$

Since P is closed in A, we get that P is complete and hence A is complete and hence A is complete in the $\|x\|_{op}$ norm.

We have $\|x\|_{op} = \|T_x\| = \sup_{\|y\|=1} \|T_x(y)\| = \sup_{\|y\|=1} \|x - y\|$

$$\geq \|x.\ \geq \left\| x.\frac{e}{\|e\|} \right\| = \frac{\|x\|}{\|e\|}$$

$\therefore$ we get that $\|x\|$ and $\|x\|_{op}$ are equivalent.

Next, we obtain

$$\|x \,.\, y\|_{op} = \|T_{xy}\| = \|T_x T_y\| \leq \|T_x\| \, \|T_y\| = \|x\|_{op} \, \|y\|_{op}$$

and since $T_e = 1$, we conclude $\|e\|_{op} = 1$.

**Note** : If $\|x \,.\, y\| \leq \|x\| \, \|y\|$ and $\|e\| = 1$, then $\|.\|_{op} = \|.\|$

**Proof** : Earlier, we have obtained that $\|x\|_{op} \geq \geq \dfrac{\|x\|}{\|e\|} = \|x\|$

Now, $\|x\|_{op} = \|T_x\| = \sup_{\|y\|=1} \|T_x(y)\|$

$= \sup_{\|y\|=1} \|x.y\|$

$\leq \sup_{\|y\|=1} \|x\| \, \|y\|$

$= \|x\|$

**Note:** The product map x.y is continuous with respect to both vairables.

Proof: Assume $x_n \to x$ and $y_n \to y$ in a Banach algebra A.

$\|x_n . y_n - x . y\| = \|x_n y_n - x_n . y + x_n . y - x . y\|$

$= \|x_n . (y_n - y) + (x_n - x) . y\|$

$\leq \|x_n\| \, \|y_n - y\| + \|x_n - x\| \, \|y\|$

$\to \|x\| . 0 + 0 . \|y\| = 0$

$\therefore x_n . y_n \to x . y$

Remark: Now onwards we assume that the conditions $\|x . y\| \leq \|x\| \, \|y\|$ and $\|e\| = 1$ are valid.

Examples :

\(i) The space of continuous functions C[0, 1] with the supremum norm and the multiplication of functions, is a Banach algebra. The identity of this algebra is the function e(t) = 1.

[$\forall t \in [0, 1]$, we have $|(x . y)(t)| = |x(t).y(t)| = |x(t)| \, |y(t)|$

$\leq \|x\| \, \|y\|$

$\Rightarrow \sup |x .y(t)| \leq \|x\| \, \|y\|$ $\quad t \in [0,1]$

$\therefore \quad \|x. y\| \leq \|x\| . \|y\|$ ]

(ii) Let X be a Banach space. The space L(X) of all bounded linear operators in X is a Banach algebra. The identity operator I is the identity of this algebra.

(iii) Consider the Banach space

$W = \{z(t); z(t) = \sum_{n=-\infty}^{\infty} C_n e^{int}, 0 \le t \le 2\pi, \|z\| = \sum_{n=-\infty}^{\infty} |c_n| < \infty\}$

The product in W is defined as the product of functions.

i.e, x(t). y(t) = z(t) on $[0, 2\pi]$ where $x = \sum a_n e^{int}, y = \sum b_n e^{int}$

and $z = \sum c_n e^{int}$ with $c_n = a_n * b_n = \sum_{m=-\infty}^{\infty} a_{n-m} b_m$

(convolution of sequences)

We have

$c_n = ..... + a_{n+2}b_{-2} + a_{n+1}b_{-1} + a_n b_0 + a_{n-1}b_1 + a_{n-2}b_2 + .....$

$|c_n| \le ..... + |a_{n+2}|\,|b_{-2}| + |a_{n+1}|\,|b_{-1}| + |a_n|\,|b_0| + |a_{n-1}|\,|b_1| + |a_{n-2}|\,|b_2| + ....$

$\therefore \|z\| \le ..... + [...+|a_0|\,|b_{-2}|+|a_{-1}|\,|b_{-1}| + |a_{-2}|\,|b_0| + |a_{-3}|\,|b_1| + ....]$

$+[..... +|a_1|\,|b_{-2}| + |a_0|\,|b_{-1}| + |a_{-1}|b_0| + |a_{-2}|\,|b_1| + ..... ]$

$+[..... +|a_2|\,|b_{-2}| + |a_1|\,|b_{-1}| + |a_0|b_0| + |a_{-1}|\,|b_1| + ..... ]$

$+[..... +|a_3|\,|b_{-2}| + |a_2|\,|b_{-1}| + |a_{-1}|b_0| + |a_0|\,|b_1| + ..... ] + .....$

$=[....+|b_{-2}| +|b_{-1}| + |b_0| + |b_{-1}|+...][....+|a_{-1}|+ |a_0|+ |a_1| + .....]$

$= \|x\|\,\|y\|$

The algebra W is called the *Weiner Algebra*. Obviously, the identity element e of this algebra is the constant function 1.

(iv) Consider the space $L_1(-\infty, \infty)$ with multiplication defined by $x * y = \int_{-\infty}^{\infty} x(t-u)\,y(u)\,du$ with this product, $L_1(-\infty, \infty)$ satisfies all of the properties of Banach algebra except the existence of an identity.

Moreover ||x * y|| ≤ ||x|| ||y||

(v) In the space $l_1$ with the usual norm define for $x=(x_1)_{i=1}^{\infty}$, $y=(y_i)_{i=1}^{\infty}$ the product $x.y=(x_i y_i)_{i=1}^{\infty}$. Then $I_1$ is a Banach algebra without identity.

$[\|xy\| = \sum_{i=1}^{\infty}|x_i y_1| = \sum_{i=1}^{\infty}|x_1||y_i| \le \|x\|_{\infty}\sum_{i=1}^{\infty}|y_i|$

$= ||x||_{\infty}||y||$

$\le ||x||\ ||y||$

If e = ($a_1$, $a_2$, .....) is an identity of this algebra, then $a_1$ = $a_2$ = ..... = 1. But this sequence does not being to $I_1$.]

(vi) Lt $k^{(n)}$ be the space of complex polynomials $x=\sum_{k=0}^{n}\alpha_k\lambda^k$ with the natural operations of sum, the product by a complex number and the usual product, with the additional condition $\lambda^{n+1}=0$, and the norm $\|x\|=\sum_{k=0}^{n}|\alpha_k|$. Then $k^{(9n)}$ is a Bnach algebra. The poloynomial e = 1 is the identity of $k^{(n)}$.

[Let

$x=\alpha_0+\alpha_1\lambda+\alpha_2\lambda^2+....+\alpha_n\lambda^n, y=\beta_0+\beta_1\lambda+\beta_2\lambda^2+....+\beta_n\lambda^n$

$\|x\|=|\alpha_0|+|\alpha_1|+|\alpha_2|+..|\alpha_n|,\ \ \|y\|=|\beta_0|+|\beta_1|+|\beta_2|+..|\beta_n|,$

$x.y=\alpha_0\beta_0+(\alpha_0\beta_1+\alpha_1\beta_0)\lambda+(\alpha_0\beta_2+\alpha_1\beta_1+\alpha_2\beta_0)\lambda^2+...$

$(\alpha_0\beta_n+\alpha_1\beta_{n-1}++\alpha_{n-1}\beta_1+\alpha_n\beta_0)\lambda^n$

$\|x.y\|=|\alpha_0\beta_0|+\alpha_0\beta_1+\alpha_1\beta_0|+|\alpha_0\beta_2+\alpha_1\beta_1+\alpha_2\beta_0$

$|+..|\alpha_0\beta_n+\alpha_1\beta_{n-1}+..\alpha_n\beta_0|\le|\alpha_0\|\beta_0|+|\alpha_0\|\beta_1|+|\alpha_1\|\beta_0|+|\alpha_0\|\beta_2|+$

$|\alpha_1\|\beta_1|+|\alpha_2|\ |\beta_0|+$

$.....+|\alpha_0\|\beta_n|+|\beta_{n-1}|+.....+|\alpha_{n-1}\|\beta_1|+|\alpha_n\|\beta_0|$

$$= |\alpha_0|\left[|\beta_0| + |\beta_1| + |\beta_2| + \ldots.. + |\beta_n|\right] + |\alpha_1|\left[|\beta_0| + |\beta_1| + |\beta_2| + \ldots..\right]$$

$$\ldots..+\left[|\beta_{n-1}|\right] + |\alpha_2|\left[|\beta_0| + |\beta_1| + |\beta_2| + \ldots.. + |\beta_{n-2}|\right] + \ldots..$$

$$\ldots. + |\alpha_{n-1}|\left[|\beta_0| + |\beta_1|\right] + |\alpha_n|\ |\beta_0|.$$

$$\leq |\alpha_0|\ \|y\| + |\alpha_1|\ \|y\| + |\alpha_2|\|y\| + \ldots|\alpha_{n-1}|\|y\| + |\alpha_n|\|y\|$$

$$= \left[|\alpha_0| + |\alpha_1| + |\alpha_2| + \ldots. + |\alpha_{-1}| + |\alpha_n|\right]\|y\|$$

$$= \|x\|\ \|y\|$$

$$x + y = (\alpha_0 + \beta_0) + (\alpha_1 + \beta_1)\lambda + (\alpha_2 + \beta_2)\lambda^2 + \ldots.. + (\alpha_n + \beta_n)\lambda^n$$

$$\|x + y\| = |\alpha_0 + \beta_0| + |\alpha_1| + |\beta_1| + |\alpha_2| + |\beta_2| + \ldots. + |\alpha_n| + |\beta_n|$$

$$\leq |\alpha_0| + |\beta_0| + |\alpha_1| + |\beta_1| + |\alpha_2| + |\beta_2| + \ldots... + |\alpha_n| + |\beta_n|$$

$$= |\alpha_0| + |\alpha_1| + |\alpha_2| + \ldots.. + |\alpha_n| + |\beta_0| + |\beta_1| + |\beta_2| + \ldots. + |\beta_n|.$$

$$= \|x\| + \|y\|.$$

For a scalar $c, c.x = c\alpha_0 + c\alpha_1\lambda + \ldots. + c\alpha_n\lambda^n$.

$$\|c.x\| = |c\alpha_0| + |c\alpha_1| + \ldots... + |c\alpha_n| = |c|\left[|\alpha_0| + |\alpha_1| + \ldots..|\alpha_n|\right] = |c|\ \|x\|.$$

Clearly ||x|| = 0 iff x = 0

$K^{(n)}$ being a finite dimentional normed space, is complete w.r.t any norm. Since the elements of $K^{(n)}$ are complex polynomials of degree atmost n, and the operations being defined naturally, the other conditions of algebra are clearly satisfied.

(vii) C is Banach algebra where the operations and norm are defined naturally.

[*i.e, the sum is the usual sum; the scalar multiplication and product are same, Norm is the absolute value function.*]

**Remark**: For the Banach spaces satisfying all of the properties of the Banach algebra except the existence of an identity, we consider the extension $A_1$, of the original Banach space $A$ where

(a) $A_1 = \{(x, \alpha), x \in A, \alpha \in C\}$

(b) the linear operations are performed coordinate wise.

(c) the product is defined by $(x, \alpha).(y, \beta) = (x . y + \alpha . y + \beta . x, \alpha. \beta)$

and

(d) the norm is $\|(x, \alpha)\| = \|x\| + |\alpha|$

This extension is a Banach algebra with identity element (0, 1)

Sometimes another notation is used: $A_1 = A \oplus \alpha e$

$[(x, \alpha).(0,1) = (x.0 + \alpha.0 + 1. x, \alpha.1) = (x, \alpha)]$

$(0, 1). (x, \alpha) = (0. x + 1. x + \alpha. 0, 1.\alpha) = (x, \alpha)$

$\|(x, \alpha) + (y, \beta)\| = \|(x+y, \alpha + \beta)\| = \|x + y\| + |\alpha + \beta|$

$\leq \|x\| + \|y\| + |\alpha| + |\beta|$

$\leq \|x\| + |\alpha| + \|y\| + |\beta|$

$= \|(x, \alpha)\| + \|(y, \beta)\|$

$\|(x, \alpha) . (y, \beta\| = \|(xy+ay + \beta x, \alpha\beta)\|$

$= \|xy + ay + \beta x\| + |\alpha\beta|$

$\leq \|x\| \ \|y\| + |\alpha| + \ \|y\| + |\beta| \ \|x\| + |\alpha| \ |\beta|$

$= \|x\| + |\alpha| + (\|y\| + |\beta|)$

$= \|(x, \alpha)\| \ \|(y, \beta)\|$

**Definition**: Let *A* be a Banach algebra. An element $x \in A$ is saaid to be invertible if there exists $x^{-1} \in A$ such that
$x.\ x^{-1} = x^{-1}.\ x = e$.

**Proposition 10.1.2**: If $\|x\| < 1$, then $e - x$ is invertible and

$$(e-x)^{-1} = \sum_{k=0}^{\infty} x^k .i.e; (e-x)^{-1} = e + x + x^2 + \ldots.$$

**Proof:** First, we note that $\|x^k\| \le \|x\|^k \to 0$ as $k \to \infty$, since $\|x\| < 1$.

$\therefore x^k \to 0$, as $k \to \infty$.

Now, consider the sequence $s_n = \sum_{k=0}^{n} x^k$ (the sequence of partial sums of $\sum_{k=0}^{\infty} x^k$)

**Claim**: $(s_n)$ is a cauchy sequence in *A*.

**Proof:** Let $m > n$. Then $s_n = e + x + x^2 + \ldots.. + x^n$ and

$s_m = e + x + x^2 + \ldots.. + x^n + x^{n+1} + \ldots...+ x^m$

$s_m - s_n = x^{n+1} + \ldots.... + x^m$

$\|s_m - s_n\| \le \|x\|^{n+1} + \ldots. + \|x\|^m \to 0$, as $m, n \to \infty$.

Hence the claim is proved.

Since *A* is complete, $(s_n)$ is convergent in *A*.$(i.e; \sum_{k=1}^{\infty} x^k$ exists in A).

Also, note that $(e - x).\ s_n = (e - x).\ (e + x + x^2 + ... + x^n)$

$= e - x^{n+1} \to e$ as $n \to \infty$.

Similarly $s_n \,.\, (e - x) \to e$ as $n \to \infty$.

$$\therefore \lim_{n\to\infty} s_n.(e-x) = e = (e-x)\lim_{n\to\infty} s_n$$

so $(e - x)^{-1} = \lim_{n\to\infty} s_n = \sum_{k=0}^{\infty} x^k$.

**Remark**: If x is invertible and $\|y\| < \frac{1}{\|x^{-1}\|}$, then x - y is invertible.

[**proof:** We have $x - y = x \,.\, e - x \,.\, x^{-1}.\, y = x \,.\, (e - x^{-1}y)$

Note that $\|x^{-1} y\| \leq \|x^{-1}\| \, \|y\| < 1$. So by above proposition $e - x^{-1} y$ is invertible $\therefore x(e - x^{-1}y) = x - y$ is invertible].

**Remark** : The set of all invertible elements of a Banach algebra $A$ is an open subset of $A$

**proof:** Let 0 be the set of all invertible elements of $A$. We show that, for any $x \in O, U\left(x, \frac{1}{\|x^{-1}\|}\right) \subset O$.

Let $y \in U\left(x, \frac{1}{\|x^{-1}\|}\right)$. Then $\|x - y\| < \frac{1}{\|x^{-1}\|}$.

$\therefore$by above remark x - (x - y) is invertible.

i.e.y is invertible. so $y \in O$.

Hence x is an interior point of $O$ and so $O$ is open.

**Result 1:** The identity element of a Banach algebra is unique.

**proof**: Let $e_1$, $e_2$ be two identity elements of the Banach algebra. Then $e_1 = e_1 e_2 = e_2$.

**Result 2:** If the element x of an algebra is invertible, then its inverse is unique.

**proof:** If possible, let $x_1$, $x_2$ be both inverses of x. Then we have $xx_1 = x_1x = e$ and $xx_2 = xx_2 = e$. Now, $x_1 = x_1e = x_1(xx_2) = (x_1x)x_2 = ex_2 = x_2$.

**Result 3**: If the elements *x* and *xy* in an algebra are invertible, then the element *y* is invertible.

**proof** : Take $z = (xy)^{-1} x$.

Then $zy = (xy)^{-1} xy = e$.

and $yz = y(xy)^{-1} x = ey(xy)^{-1} x = (x^{-1}x) y(xy)^{-1}x = x^{-1}(xy(xy)^{-1}x = x^{-1} ex = x^{-1} x=e$.

**Result 4:** If the elements *xy* and *yx* in an algebra are invertible then the elements *x* and *y* are invertible.

**proof:** Take $z = (xy)^{-1} x$

Then $zy = (xy)^{-1} xy = e$, and

$yz = y(xy)^{-1} x = (yx)^{-1} (yx)y(xy)^{-1} x = (yx)^{-1} y(xy)(xy)^{-1} x=(yx)^{-1} yex = (yx)^{-1}yx=e$. So we get that y is invertible and $y^{-1} = (xy)^{-1}x$.

Now, take $v = (yx)^{-1}y$.

Then $vx = (yx)^{-1} yx = e$, and

$xv = x(yx)^{-1} y= (xy)^{-1} (xy)x(yx)^{-1} y= (xy)^{-1} x(yx) (yx)^{-1}y=(xy^{-1}xey = (xy)^{-1} xy = e$

so x is invertible and $x^{-1} = (yx)^{-1} y$.

Result 5: If the element e - yx of an algebra is invertible, then the element e - xy is invertible.

proof: Take $z = e + x(e-yx)^{-1}y$.

Then $z(e - xy) = [e + x(e - yx)^{-1} y] [e - xy]$

$= e - xy + x(e - yx)^{-1} y - x (e - yx)^{-1} yx\ y$

$= e - xy + x(e - yx)^{-1} y - x (e - yx)^{-1} (yx - e + e)y$

$= e - xy + x(e - yx)^{-1} y - x (e - yx)^{-1} (yx - e) y - x(e - yx)^{-1}\ ey$

$= e - xy + x(e - yx)^{-1} y + xy - x(e - yx)^{-1} y = e.$

$(e - xy)z = (e - xy) [e + x (e - yx)^{-1} y]$

$= e - xy + x(e - yx)^{-1} y - xyx (e - yx)^{-1}$

$= e - xy + x(e - yx)^{-1} y - x(yx - e + e)(e - yx)^{-1}y$

$= e - xy + x(e - yx)^{-1} y - x(yx - e )(e - yx)^{-1}y - xe (e - yx)^{-1} y$

$= e - xy + x(e - yx)^{-1} y + xy - x(e - yx)^{-1}y$

$= e$

**Result 6**: If x is an element of an algebra and $||x|| < 1$, then

$$\left\|(e-x)^{-1} - e - x\right\| \le \frac{\|x\|^2}{1-\|x\|}.$$

**proof** : By proposition 10.1.2, we have $(e - x)^{-1} = e + x + x^2 + x^3 + ....$

$\therefore \|(e - x)^{-1} - e - x\| = \|x^2 + x^3 + .....\|$

$\le \|x\|^2 + \|x\|^3 + .....$

$= \frac{\|x\|}{1-\|x\|}$\

**Result 7:** If x is an invertible elements of an algebra and

$\|x-y\| \le \frac{1}{\left\|x^{-1}\right\|}$, then y is invertible and

$$\left\|x^{-1}-y^{-1}\right\| \leq \frac{\left\|x^{-1}\right\|^2\|x-y\|}{1-\left\|x^{-1}\right\|\|x-y\|}$$

**Proof**:

We have $y = x + (y - x) = xe + x\,x^{-1}(y - x) = x[e + x^{-1}(y - x)]$

we note that $\|x^{-1}(y - x)\| \leq \|x^{-1}\|\,\|y - x\| < \|x^{-1}\|$. $\dfrac{1}{\left\|x^{-1}\right\|} = 1.$

$\therefore e + x^{-1}(y - x)$ is invertible.

So the element y is invertible and $y^{-1} = [e + x^{-1}(y - x)]^{-1}\,x^{-1}$.

We have $\left[e+x^{-1}(y-x)\right]^{-1} = e+\sum_{k=1}^{\infty}\left[x^{-1}(y-x)\right]^k$

$$\therefore \left[e+x^{-1}(y-x)\right]^{-1} - e = \sum_{k=1}^{\infty}\left[x^{-1}(y-x)\right]^k$$

Now, $y^{-1}-x^{-1} = \left[e+x^{-1}(y-x)\right]^{-1}x^{-1} - x^{-1}$

$$= \left\{\left[e+x^{-1}(y-x)\right]^{-1} - e\right\}x^{-1}$$

$$= \left\{\sum_{k=1}^{\infty}\left[x^{-1}(y-x)\right]^k\right\}x^{-1}$$

$$= \sum_{k=1}^{\infty}\left(x^{-1}\right)^k(y-x)^k\,x^{-1}$$

$$\therefore \left\|y^{-1}-x^{-1}\right\| \leq \sum_{k=1}^{\infty}\left\|x^{-1}\right\|^k\left\|y-x\right\|^k\left\|x^{-1}\right\|$$

$$= \frac{\left\|x^{-1}\right\|^2\|y-x\|}{1-\left\|x^{-1}\right\|\|y-x\|}$$

**Definition 10.1.4:** A subset $I$ of a commutative Banach algebra $A$ is called an ideal if and only if $I$ is a subspace of $A$ and for any $x \in \mathrm{I}, \mathrm{y} \in \mathrm{A}$ we have $\mathrm{x}\,\mathrm{y} \in \mathrm{I}$.

The ideal $I$ is called a proper ideal if $I$ is not equal to (0) or A.

We make a few observations regarding Commutative Banach Algebras.

(i) If the inverse of xy exists, then both x and y are invertible.

**Proof:** Take $z = y\,(xy)^{-1}$

Then $xz = xy\ (zy)^{-1} = \mathrm{e}$

$\therefore$ $z$ is invertible

Next, take $v = x\,(yx)^{-1}$

Then $yv = yx\,(yx)^{-1} = e$. $\therefore$ $y$ is invertible

(ii) $x$ is invertible if and only if $x$ does not belong to any proper ideal. [$x$ is assumed to be nonzero].

**Proof:** Let $x$ be invertible. If possible, assume that $x \in \mathrm{I}$, where $I$ is a proper ideal.

Then since $x^{-1} \in A$ we get that $\mathrm{e} = \mathrm{x}^{-1}x \in I$.

So we get $I=A$, which is a contradiction since $I$ is proper.

Next, assume that $x$ is not invertible. Consider $I=xA = \{x\,y;\ y \in A\}$

Then $I$ is clearly an ideal

$[xy_1 + xy_2 = x(y_1 + y_2) \in I; \alpha(xy) = x(\alpha y) \in I$

For $z \in A$ and $\mathrm{x}\,\mathrm{y} \in \mathrm{I}$, we have $\mathrm{x}\,(\mathrm{xy}) = \mathrm{x}(\mathrm{zy}) \in \mathrm{I}]$

clearly we have $e \notin I$.

[If possible let $e \in I$, Then $e = x.y$ for somes $y \in A$

$\therefore$ $x$ is invertible, which is not true by our assumption].

$\therefore$ $I$ is a proper ideal of $A$.

We not that $x = x.e \in I$,

*i.e.x* belongs to a proper ideal.

(iii) If $I$ is a proper ideal, then its closure $\bar{I}$ is also a proper ideal.

**Proof:** The fact that $\bar{I}$ is an ideal can be proved using the continuity of addition, scalar multiplication and product in Banach algebra.

[Let $x, y \in \bar{I}$ . Then $\exists$ sequences $(x_n)$. $(y_n)$ in $I$ such that $x_n \to x$ and $y_n \to y$.

$\therefore$ $x_n + y_n \to x + y$ Since $x_n + y_n \in I$, we get that $x + y \in \bar{I}$

Now, we have $\alpha x_n \to \alpha x$.

Since $\alpha x_n \in I$, *we get* $\alpha x \in \bar{I}$

Now, let $y \in \bar{I}$ and $x \in A$. We have a sequence $(y_n)$ in I s.t. $y_n \to y$.

Since the product is continuous w.r.t the first variable, we get $x\, y_n \to x.\, y.$ since $x\, y_n \in I$, we get $xy \in \bar{I}$]

**Claim:** *U (e,1),* the open ball of radius 1 around *e* does not intersect with I.

**Proof:** Let $y \in U$ (e, 1). Then $\|e - y\| < 1 = \dfrac{1}{\|e^{-1}\|}$

$\therefore$ by remark 10.1.3, $e$ –($e$–$y$) is invertible] [or use 10.1.2 since ||e-y||<1, e-(e-y) is invertible.

i.e. $y$ is invertible.

$\therefore$ by the above property, $y \notin I$, since $I$ is a proper ideal so our claim is proved.

$\therefore$ we get that no element of $U$ (e, 1) is a limit point of $I$, also no element of $U$ (e, 1) is in $I$].

$\therefore$ no element of $U$ (e, 1) is in $\bar{I}$

$\therefore \bar{I}$ is a proper ideal.

**Corollary 10.1.5:** If $A$ does not have any proper ideal, then $A$ is a field. i.e; every non-zero element is invertible.

**Proof:** Let $x$ be a non-zero element of $A$.

Since $A$ does not have any proper ideal, we get that $x$ does not belong to any proper ideal. $\therefore$ by property (ii) above, we get that $x$ is invertible.

**Definition** Let $A$ be a commutative Banach Algebra. A proper ideal $M \subseteq A$ is called a maximal ideal if there is no proper ideal which contains the ideal $M$.

**Note:** If $M$ is a maximal ideal, then $M$ is closed.

**Proof:** If possible, assume that $M$ is not closed. i.e $M \neq \overline{M}$. Then $\overline{M}$ will be a proper ideal containing $M$, which is a contradiction since $M$ is maximal.

$\therefore$ $M$ must be closed.

**Example:** Consider the algebra $C$ [$0,1$]

For a fixed $a \in [0,1]$, let $M_a = \{x;\ x(a)=0\}$.

**Claim 1:** $M_a$ is an ideal of $C$ [0,1].

[**Proof:**

$$x, y \in \mathrm{M_a} \Rightarrow x(a) = 0, y(a) = 0 \Rightarrow (x+y)(a) = x(a) + y(a) = 0$$

$$(ax)(a) = \alpha.x(a) = \alpha.0 = 0$$

Let $x \in M_a, y \in C[0,1]$. Then $(y.x)(a) = y(a).x(a) = 0]$

**Claim 2:** Codim $M_a$ =1.

[Consider the function $f(t) = I, \forall t \in [0,1]$.

**Claim:** $\{[f]\}$ is a basis for $C[0,1]/M_a$

**Proof:** Let

$\alpha[f] = [0] \Rightarrow [\alpha f] = [0] \Rightarrow \alpha f \in M_a \Rightarrow [\alpha f][\alpha] = 0 \Rightarrow$

$\alpha.f(a) = 0 \Rightarrow \alpha.1 = 0 \Rightarrow \alpha = 0.$

Let [x] $\in$C[0, 1]/Ma

$[x] = \alpha[f] \Rightarrow [x] = [af] \Rightarrow x - \alpha f \in M_a \Rightarrow x(a) - \alpha.f(a) = 0 \Rightarrow$

$x(a) = \alpha \therefore [x] = x(a)[f]]$

$\therefore M_a$ is a maximal ideal of $C[0,1]$.

[Let $U \hookrightarrow W$ and *codim* $U$ =I. Then $U$ is a maximal subspace.

**Proof:** Let $U \subset V \subseteq W$

Let $\{[W]\}$be a basis for *W/U*

**Claim:** $w \in V$;

**Proof:** For any $v \in V$, we have $[v] = \alpha[w]$

$\Rightarrow [v] = [\alpha w] \Rightarrow v - \alpha w \in U$

$\therefore v - \alpha w \in V$

Since $v \in V$, we get $\alpha w \in V$ and hence $w \in V$.

$\therefore$ our claim is proved.

Next, we show that $V = W$

Let $p \in W$. Then $[p] = \beta[w] \Rightarrow p - \beta w \in U \Rightarrow p - \beta w \in V$.

Since $w \in V$, we have $\beta w \in V$. $\therefore$ we get $p \in V$ So $W \subseteq V$. Hence $W=V$ Thus $U$ is a maximal subspace]

Next, we show that every maximal ideal of C [0,1]is of he form $M_a$ for some $a \in$ [0.1].

Let M be a maximal ideal of c[0,1].

**Claim:** $\exists$ $a \in [0, 1]$ such that $x(a)=0$, for all $x \in$M.

**Proof:** Assume the contray

i.e, for all $a \in [0,1]$, $\exists$ $x_a \in M$ such that $x_a(a) \neq 0$. Since $x_a$ is a continuous function, there is an open interval $I_a$ around a such that $x_a(t) \neq 0, \forall t \in I_a$.

So we have a covering $\{I_a\}_{a \in [0,1]}$ *of* [0,1].

since [0,1] is compact, we can take a finite sub covering $\left(I_{a_1}, I_{a_2}, \ldots\ldots I_{a_n}\right)$

Consider the function

$$x(t) = x_{a_1}(t)\Big|^2 + \Big|x_{a_2}(t)\Big|^2 + \ldots\ldots + \Big|x_{a_n}(t)\Big|^2 (t)^2, \therefore t \in [0,1].\Big|$$

clearly $x(t) \neq 0, \forall t \in [0.,1]$.

$\left(Let\ t \in [0,1].\ Then\ t \in I_{a_k}\ for\ some\ k = 1,\ldots n.\ Then\ x_{a_k} \neq 0. \therefore x(t) \neq 0\right)$

Thus x is invertible and $x \in M$.

[ $y(t) = \dfrac{1}{x(t)}, t \in [0,1]$ is the inverse of $x$.

Infact $x = x_{a_1}.x_{a_1}. + x_{a_2}.x_{a_2} + \ldots\ldots + x_{a_n}.x_{a_n}.$

Each $x_{a_k} \in M$ and M being an ideal, $x \in M$ ]

Since an invertible element does not belong to any proper ideal, we get a contradiction and hence the claim is proved.

$]\left[x \in M \Rightarrow x(a) = 0 \text{ (by the claim)} \Rightarrow x \in M_a\right]$

But the ideal $M$ is maximal. $\therefore$ M = $M_a$

## 10.2 Gelfand theorem on maximal ideals

**Theorem 10.2.1:** For every proper ideal $I \subseteq A$, there exists a maximal ideal $M$ such that $I \subseteq A$.

i.e; every proper ideal is contained in a maximal ideal.

**Proof:** We use Zorn's lemma. Consider the set $\mathcal{B}$ of all proper ideals $J \subseteq I$, there exists a maximal ideal M such that $I \subseteq I$.

*i.e.*, $\mathcal{B} = \{J; I \subseteq J$ and $J$ is a proper ideal$\}$. Define a partial order $\leq$ on $\mathcal{B}$ as $J_1 \leq J_2$ iff $J_1 \subseteq J_2$

Let $\mathcal{F} = \{J_i\}$ be a linear chain.

i.e., For all $J_1, J_2 \in \mathcal{F}$ either $J_1 \leq J_2$ or $J_2 \leq J_1$.

Consider $\tilde{J} = \bigcup J_i$.

Since the chain is linear, $\tilde{J}$ is a subspace. Let $x \in \tilde{J}$ and $y \in A$. Then $x \in J_i$ for some $i$. $\therefore yx \in \tilde{J}$, since $\tilde{J}$, is an ideal. So $yx \in \bigcup \tilde{J} = J_i$. $\therefore \bar{J}$ is an ideal. since $e \notin J_i$ for all $i$, We get $e \notin J$. So $\bar{J}$ is a proper ideal.

Thus we get that $\bar{J}$ is an upperbound for the chain $\mathcal{F}$. Consequently, by Zorn's lemma $\mathcal{B}$ has a maximal element. ie., $\exists$ a maximal ideal containing $I$.

**Corollary 10.2.2:** An element $x$ is invertible iff $x$ does not belong to any maximal ideal.

**Proof:** Let $x$ be invertible. Then $x$ does not belong to any proper ideal. Since a maximal ideal is proper, We get that $x$ does not belong to any maximal ideal.

Now let $x$ be not invertible. Then $x$ belongs to some proper ideal. By above theorem every ideal is contained in a maximal ideal. $\therefore$ $x$ belongs to some maximal ideal.

**Quotient Algebra:** Let $I$ be a closed ideal of a Commutative Banach Algebra A. (assume I is proper). In the Banach space $A/I$ define the product $[x]\,.[y] = [x.y]$. Then $A/I$ is a Banach Algebra.

**Proof:** Since $(x+I)\ (y+I) \subseteq xy + I$, the product is well-defined.

[*Let* $[x]=[x_1]$ and $[y]=[y_1]$. Then $x-x_1 \in I$ and $y-y_1 \in I$. $\therefore x-x_1=i_1$ and

$y-y_1=i_2$ for some $i_1, i_2 \in I$. *i.e.*$x=x_1+i_1$ and $y=y_1+i_2$. $\therefore xy=(x_1+i_1)$.

$(y_1+i_2)=x_1y_1+x_1i_2+i_1y_1+i_1i_2=x_1+y_1+i_3,$ for some $i_3 \in I$.

$\therefore [xy]=[x_1y_1]]$

For any $\in > 0$ take $z_1z_2 \in I$ such that $\|x+z_1\|<\|x\|+\in$ and $\|y+z_2\|<\|[y]\|+\in$

[Such a choice of $z_1 \in I$ us always possible, because

$\|[x]\| =$ in $\{f\|x+z\|, z \in I\}$. Similarly we can choose $z_2$ aswell]

Now,

$\|[xy]\| \le \|xy+z_1y+z_2x+z_1.z_2\|$, since $z_1y+z_2x+z_1z_2 \in I$

$= \|(x+z_1)(y+z_2)\|$ (1)

$$= \|x + z_1\| \|y + z_2\|$$

$$= [\|(x)\| + \in] \, \|[y]\| + \in]$$

This holds for every $\in > 0$. $\therefore$ by letting $\in \to 0$ , we get $\|[xy]\| \le \|x\| \, \|[y]\|$. Clearly $[e]$ is the identity element of *A/I*.

Next, we show that $\|[e]\| = 1$. Clearly, we have $\|[e]\| \le \|e + 0\| = \|e\| = 1$.

*Claim:* For any $u \in I, \|e + u\| \ge 1$.

*Proof:* If possible assume that $\exists \; u \in I$ such that $\|e + u\| < 1$. Then we get $-u = e - (e + u)$ is invertible, which is a contradiction (invertible elements do not belong to proper ideals. $\therefore$ the claim is proved.

Thus we get that $\|[e]\| = in\, f\{\|e + u\|; u \in I\} \ge 1$. The other properties follow directly from the definition.

**Result:** Let *A* be a Commutative Banach algebra and let *I* be an ideal of *A*. Then the following are true.

(i) If $I \subseteq J \subseteq A$ and *J* is an ideal of *A*, then *J/I* is an ideal of *A/I*.

ii) Every ideal in *A/I* has the form *J/I*, where $J \supseteq I$ is an ideal of *A*.

(iii) If $I \subseteq J \subseteq A$ and *J* is an ideal of *A*, then *J* is proper iff *J/I* is a proper ideal of *A/I*.

**Proof:**

**(i)** Let $[x], [y] \in J/I$. So $x, y \in J$

$\therefore x + y \in J$ and $\alpha x \in J$ since *J* is an ideal.

So $[x+y] \in J/I$ and $[\alpha x] \in J/I$.

i.e; $x + y \in J/I$ and $\alpha [x] \in J/I$. and hence $J/I$ is a subspace.

Now let $[x] \in J/I$ and $[z] \in A/I$.

$\therefore x \in J$ and $z \in A$.

So $xz \in J$, since $J$ is an ideal.

$\therefore [xz] \in J/I$ i.e. $[x].[z] \in J/I$

Thus, $J/I$ is an ideal of $A/I$.

**(ii)** Let $B$ be an ideal of $A_1$. Set $J = \{u; [u] \in B\}$.

*Claim:* $I \subseteq J$.

*Proof:* Let $y \in I$, $\therefore [y]$ is the zero element of $A/I$.

Since $B$ is an ideal of $A/I$, $[y] \in B$. $\therefore y \in J$. So the claim is proved.

Since $B$ is an subspace of $A/I$, clearly $J$ is a subspace of $A$.

$[x, y \in J \Rightarrow [x], [y] \in B \Rightarrow [x + y] \in B \Rightarrow x + y \in J$.
$x \in J \Rightarrow [x], \in B \Rightarrow [\alpha x] \in B \Rightarrow \alpha x \in J$

Now let $x \in J$ and $z \in A$.

Then $[x] \in B$ and $[z] \in A/I$.

Since $B$ is an ideal, we get $[xz] \in B$.

$\therefore$ We get $xz \in J$, by the construction of $J$.

Hence $J$ is an ideal. It is obvious that B = J/I

$[ [u] \in B \Leftrightarrow u \in J \Leftrightarrow [u] \in J/I ]$.

**(iii)** Assume that $J$ is proper. Then $\notin J$. This clearly gives $[e] \notin J/1$.

[If possible, $[e] \in J/I$. Then $e=x+y$, for some $x \in J$, $y \in I \subseteq J$. This gives $e \in J$, which is a contradiction].

Hence the ideal $J/I$ is proper.

Conversely, let $J/I$ be a proper ideal of $A/I$.

Then $[e] \notin J/I$ and hence $e \notin J$.

$\therefore$ we get that $J$ is proper.

**Corollary 10.2.3:** An ideal $M$ of $A$ is maximal iff $A/M$ is a field.

**Proof:** Let $M$ be a maximal ideal of $A$.

$\therefore$ there are no proper ideals $J$ such that $M \subseteq J \subseteq A$. So by the above result (iii) there are no proper ideals for $A/M$. $\therefore$ By corollary 10.1.5, we get that $A/M$ is a field.

Conversely assume that $A/M$ is a field.

$\therefore$ $A/M$ has no proper ideals. [every non zero element of $A/M$ is invertible, and an invertible element does not belong to any proper ideal.]

$\therefore$ Again by the above result (iii), there is no proper ideal $J$ such that $M \subseteq J \subseteq A$. Hence $M$ is a maximal ideal of $A$.

**1)** Let $A$ be a Banach algebra and let $\phi : A \to C$ be a non-zero multiplicative linear functional.

i.e;
$\varphi$ is non-zero, linear and $\varphi(x.y) = \varphi(x)\ \varphi(y)$, for any $x, y \in$ A. Then:

(a) $\phi(e) = 1$ and $\phi(x) \neq 0$ for any invertible $x$.

(b) If $x \in A$ and $\|x\| < 1$, then $|\phi(x)| < 1$.

(c) $\phi$ is bounded and $\|\phi\| = 1$.

**Proof:**

(a) Take $x_0 A$ such that $\varphi(x_0) \neq 0$.

Then we have $\phi(x_0) \neq \phi(x_0.e) = (x_0).\phi(e)$

$\therefore$ We get $\phi(e) = 1.$

If $x \in A$ is invertible, then we have $xx^{-1} = e.$

$$\therefore \phi(xx^{-1}) = \phi(e)$$

$$i.e; \phi(x)\phi(x^{-1}) = 1$$

$$\therefore \phi(x) \neq 0.$$

(b) *Claim*: If $\lambda \in C$ and $|\lambda| \geq 1$, then $e - \lambda^{-1}x$ is invertible.

*Proof:* We have $\|\lambda^{-1x}\| = \frac{1}{|\lambda|}\|x\| < 1$, since $|\lambda| \geq 1$ and $\|x\| < 1.$

$\therefore$ The element $e - \lambda^{-1}x$ is invertible. So the claim is proved.

Since $e - \lambda^{-1}.x$ is invertible, by (a) we get $\phi(e - \lambda^{-1}x) \neq 0.$

i.e; $\phi(e) - \lambda^{-1}\phi(x) \neq 0 \Rightarrow 1 - \frac{1}{\lambda}\phi(x) \neq 0 \Rightarrow \phi(x) \neq \lambda.$

Hence, we get that $|\phi(x)| < 1.$

(c) From (b), We get that $\phi$ is bounded and $\|\phi\| \leq 1.$

[For every $x \neq 0$, we have $\left\| \frac{x}{2\|x\|} \right\| = \frac{1}{2} < 1.$

$$\therefore by(b), \left| \phi\left( \frac{x}{2|x|} \right) \right| < 1 \Rightarrow |\phi(x)| < 2\|x\|, \forall x.$$

So $\phi$ is bounded. Also $\|\phi\| = \sup_{\|x\|<1} |\phi(x) \leq 1]|$

Since $\phi(e)=1$, we conclude that $\|\phi\|=1$

2. Consider the Banach algebra $l_1$.

Let $M=\left\{x=(x_n)_{n=1}^{\alpha}\in l_1; x_1=0\right\}$ Show that *M* is a maximal ideal.

[The fact that *M* is an ideal is a direct verification. Since Codim *M*=1, We get that *M* is a maximal ideal].