REPRESENTATION THEORY

(MTH4E12)

SELF LEARNING MATERIAL

IV SEMESTER

M.Sc. MATHEMATICS
(2019 Admission)

UNIVERSITY OF CALICUT

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MTH4E12 : Representation Theory

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Preface

The Self Learning Material (SLM)- MTH4E12: REPRESENTATION THEORY is prepared based on the syllabus for M.Sc. Mathematics (CBCSS) PG Programme of University of Calicut effective from 2019 admission onwards. The material is mainly intended for helping the students who are studying M.Sc. Mathematics course under the School of Distance of Education, University of Calicut. The material is prepared based on the text book, Introduction to Group Characters (Second Edition) by Walter Ledermann (Cambridge University Press).

This course provide a straightforward introduction to the characters of a finite group over the complex field. The only prerequisites are a knowledge of the standard facts of Linear Algebra and a modest acquaintance with group theory. In the first chapter, the basic ideas of group representations are presented and all the preliminaries are stated. The students learn Maschke's Theorem and Schur's Lemma which are claimed to be the two pillars of Representation Theory. In the second chapter, various properties of group characters are given in detail. Third chapter deals with properties of induced characters and fourth chapter deals with permutation groups. All the main results in the book are stated in both matrix language and module language. The matrix approach lends itself more readily to computation, while the use of modules make the theory more elegant.

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Chapter 1

Group Representations

Structure of a group does not significantly depend upon the nature of its elements. Thus a concrete group is isomorphic to an abstract group. When we reverse this process of abstraction by considering homomorphisms $\theta : G \to \Gamma$, where G is an abstract group and Γ is one of the concrete groups, such a homomorphism is called a representation of G. Thus representations may be permutations, matrices, linear transformations etc.

Example for a permutation representation:

Cayley's theorem: Every group is isomorphic to a permutation group.

A finite group $G : x_1(= 1), x_2, \ldots, x_g$ can be represented as a group of permutations of degree g, that is by permutations acting on g objects. In this case, the objects are the elements of G themselves. With a typical element x of G we associate

the permutation,

$$\pi(x) = \begin{pmatrix} x_1 & x_2 & \dots & x_g \\ x_1x & x_2x & \dots & x_gx \end{pmatrix}.$$

Thus, briefly $\pi(x) : x_i \to x_ix$ for $i = 1, 2, \dots, g$. If y is another element of G,

$$\pi(y) = \begin{pmatrix} x_1 & x_2 & \dots, & x_g \\ x_1y & x_2y & \dots & x_gy \end{pmatrix}$$

That is, $\pi(y) : x_i \to x_iy$ for $i = 1, 2, \dots, g$. Thus $\pi(x)\pi(y)$ signifies the operation whereby a typical element x_i of G is first multiplied by x and then by y on the right, that is $\pi(x)\pi(y) : x_i \to (x_ix)y$ $(i = 1, 2, ., g)$. Also, we have obtained $\pi(x)\pi(y) = \pi(xy)$ which means that $\pi : G \to S_g$ is a homomorphism of G into the symmetric group S_g, the group of all permutations on g symbols. This homomorphism, which is the content of Cayley's Theorem, is called the right-regular representation of G.

A given group G may have more than one representation by permutations, possibly of different degrees. Suppose that H is a subgroup of G of finite index n, and let $G = Ht_1 \cup Ht_2 \cup \cdots \cup Ht_n$ be the coset decomposition of G relative to H. With a typical element x of G we associate the permutation, $\sigma(x) = \begin{pmatrix} Ht_1 & Ht_2 & \dots & Ht_n \\ Ht_1x & Ht_2x & \dots & Ht_nx \end{pmatrix}$, in which the permuted objects are the n cosets.

$\sigma(x) : Ht_i \to Ht_ix$ $(i = 1, 2, \dots, n)$, $\sigma(y) : Ht_i \to Ht_iy$ $(i = 1, 2, \dots, n)$.

$\sigma(x)\sigma(y) : Ht_i \to (Ht_i x)y$ $(i = 1, 2, \ldots, n)$. But, by definition $\sigma(xy) : Ht_i \to Ht_i(xy)$ $(i = 1, 2, \ldots, n)$. This proves that the map $\sigma : G \to S_n$ is a homomorphism of G into S_n.

1.1 Matrix Representations

Definition 1. *Suppose that with each element x of the group G there is associated an m by m non-singular matrix $A(x) = (a_{ij}(x))$ $(i, j = 1, 2, ..., m)$, with coefficients in the field K, in such a way that*

$$A(x)A(y) = A(xy)(x, y \in G). \tag{1.1}$$

Then $A(x)$ is called a ***matrix representation*** *of G of degree (dimension) m over K.*

Put $x = y = 1$ in (1.1). Then we have that $A(1)^2 = A(1)$. Since $A(1)$ is non-singular, it follows that $A(1) = I$, the unit matrix of dimension m. Next, put $y = x^{-1}$. Then $A(x)A(x^{-1}) = I$, so that $A(x^{-1}) = (A(x))^{-1}$.

A representation A need not be injective. The kernel of A consists of those elements u of G for which $A(u) = I$. The kernel is always a normal subgroup of G.

First, to see that the kernel $K = \{u \in G : A(u) = I\}$ is a subgroup of G.

As $A(1) = I$, $1 \in K$. Thus $K \neq \emptyset$.

Let $u, v \in K$. Then $A(u) = I$, $A(v) = I$, and so $A(v^{-1}) = (A(v))^{-1} = I$. Hence $A(uv^{-1}) = A(u)A(v^{-1}) = I^2 = I$. So we get $uv^{-1} \in K$ and hence K is a subgroup of G.

To show that K is normal, we will show that $xKx^{-1} \subseteq K$ for $x \in G$.
Let $u \in K$. Consider xux^{-1} for $x \in G$.
$A(xux^{-1}) = A(x)A(u)A(x^{-1}) = A(x)1(A(x))^{-1} = I$ for all x in G.
$\implies$ $xux^{-1} \in K$ and hence $xKx^{-1} \subseteq K$.
Thus we have obtained that K is a normal subgroup of G.
The representation is injective or faithful if and only if the kernel reduces to the trivial group $\{1\}$.
To see this, first suppose that the kernel is $\{1\}$.
For $x, y \in G$, let $A(x) = A(y)$.
This is equivalent to $A(x)(A(y))^{-1} = A(xy^{-1}) = I$ and for a faithful representation this implies that $xy^{-1} = 1$, that is $x = y$.
Next, suppose that A is injective. Consider an element x in the kernel. Then $A(x) = I$. As $A(1) = I$, we get $x = 1$, and hence the kernel reduces to the trivial group $\{1\}$.

1.1.1 Linear Representations

When $m = 1$, the representation is said to be linear. Thus a linear representation is a function on G with values in K, say $\lambda : G \to K$ such that $\lambda(x)\lambda(y) = \lambda(xy)$.
Every group possesses the trivial (principal) representation given by the constant function $\lambda(x) = 1$ for all $x \in G$.
A non-trivial example of a linear representation is furnished by the alternating character of the symmetric group S_n, (for

each $n > 1$). This is defined by

$$\zeta(x) = \begin{cases} 1 & \text{if } x \text{ is even} \\ -1 & \text{if } x \textit{ is} \text{ odd} \end{cases}$$

($\zeta(x)$ is to be read as zeta(x))

Then we have $\zeta(x)\zeta(y) = \zeta(xy)$.

Let $A(x)$ be a representation of G and suppose that $B(x) = T^{-1}A(x)T$, where T is a fixed non-singular matrix with coefficients in K. Then we can verify that $B(x)B(y) = B(xy)$, so that $B(x)$ is also a representation of G. We say that the representations $A(x)$ and $B(x)$ are equivalent over K, and we write $A(x) \sim B(x)$.

1.2 G-modules

Consider a linear map $\alpha : V \to W$, where V and W are vector spaces over K. The image of a vector $\mathbf{v}$ under α will be denoted by $\mathbf{v}\alpha$. Since α is linear, for all $\mathbf{u}, \mathbf{v} \in V$ and $h, k \in K$,

$(h\mathbf{u} + k\mathbf{v})\alpha = h\mathbf{u}\alpha + k\mathbf{v}\alpha$.

The zero map, denoted by 0, is defined by $\mathbf{v}0 = \mathbf{0}$ for all $\mathbf{v} \in V$.

We consider only finite-dimensional vector spaces.

Let $\dim V = m$, $\dim W = n$, and write

$$V = [\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m], W = [\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n] \tag{1.2}$$

to express that $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m$ and $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$ are bases of V and W respectively.

The image of $\mathbf{p}_i$ under α is some vector in W and therefore a linear combination of the basis vectors of W. Thus we have a system of equations

$$\mathbf{p}_i\alpha = \sum_{j=1}^{n} a_{ij}\mathbf{q}_j \quad (i = 1, 2, \ldots, m), \tag{1.3}$$

where $a_{ij} \in K$.

So by the principle of linearity, if $\mathbf{v} = \sum_{i=1}^{m} h_i\mathbf{p}_i$, then $\mathbf{v}\alpha = \sum_{i=1}^{m} h_i\mathbf{p}_i\alpha = \sum_{i=1}^{m}\sum_{j=1}^{n} h_i a_{ij}\mathbf{q}_j$.

Hence the $m \times n$ matrix $A = (a_{ij})$ describes the linear map α relative to the bases (1.2).

If we had used different bases, say

$$V = [\mathbf{p}'_1, \mathbf{p}'_2, \ldots, \mathbf{p}_m], W = [\mathbf{q}'_1, \mathbf{q}'_2, \ldots, \mathbf{q}'_n] \tag{1.4}$$

the same linear map α would have been described by the matrix $B = (b_{\lambda\mu})$, whose coefficients appear in the equations

$$\mathbf{p}'_\lambda\alpha = \sum_{\mu=1}^{n} b_{\lambda\mu}\mathbf{q}'_\mu \quad (\lambda = 1, 2, \ldots, m) \tag{1.5}$$

The change of bases is expressed by

$$\mathbf{p}_i = \sum_{\lambda=1}^{m} t_{i\lambda}\mathbf{p}'_\lambda \quad (i = 1, 2, \ldots, m)$$

$$\mathbf{q}_j = \sum_{\mu=1}^{n} s_{j\mu}\mathbf{q}'_\mu \quad (j = 1, 2, \ldots, n)$$

where $T = (t_{i\lambda})$ and $S = (t_{j\mu})$ are non-singular matrices of dimensions m and n respectively.

Inverting the first set of equations above, we get

$$\mathbf{p}'_\lambda = \sum_{i=1}^{m} \tilde{t_{\lambda i}} \mathbf{p}_i \quad (\lambda = 1, 2, \ldots, m)$$

where $T^{-1} = (\tilde{t_{\lambda i}})$. Hence

$$\begin{aligned}\mathbf{p}'_\lambda \alpha &= \sum_{i=1}^{m} \tilde{t_{\lambda i}} \mathbf{p}_i \alpha \quad (\lambda = 1, 2, \ldots, m) \\ &= \sum_{i,j} \tilde{t_{\lambda i}} a_{ij} \mathbf{q}_j \\ &= \sum_{i,j,\mu} \tilde{t_{\lambda i}} a_{ij} s_{j\mu} \mathbf{q}'_\mu.\end{aligned}$$

Also, from (1.5),

$$\begin{aligned}\mathbf{p}'_\lambda \alpha &= \sum_{\mu=1}^{n} b_{\lambda\mu} \mathbf{q}'_\mu \quad (\lambda = 1, 2, \ldots, m) \\ &= \sum_{i,j} \tilde{t_{\lambda i}} a_{ij} \mathbf{q}_j \\ &= \sum_{i,j,\mu} \tilde{t_{\lambda i}} a_{ij} s_{j\mu} \mathbf{q}'_\mu.\end{aligned}$$

Thus we have

$$\sum_{\mu=1}^{n} b_{\lambda\mu} \mathbf{q}'_\mu = \sum_{i,j,\mu} \tilde{t_{\lambda i}} a_{ij} s_{j\mu} \mathbf{q}'_\mu$$

which gives $B = T^{-1}AS$.

Now, suppose that $V = W$ and α is invertible. Such a linear map $\alpha : V \to V$ is called an automorphism of V over K. The matrix which describes α relative to any basis is non-singular; and any two matrices A and B which express α relative to two different bases are connected by an equation of the form $B = T^{-1}AT$. The set of all automorphisms of

V over K forms a group which we denote by $\mathscr{A}_K(V)$, or simply by $\mathscr{A}(V)$. If α_1 and α_2 are two elements of $\mathscr{A}(V)$, their product $\alpha_1\alpha_2$ is defined by operator composition, that is, if $\mathbf{v} \in V$, then $\mathbf{v}(\alpha_1\alpha_2) = (\mathbf{v}\alpha_1)\alpha_2$. We now consider representations of G by automorphisms of a vector space V. Thus we are interested in

$$G \to \mathscr{A}_K(V). \tag{1.6}$$

That is, with each element x of G, there is associated an automorphism $\alpha(x) : V \to V$ in such a way that $\alpha(x)\alpha(y) = \alpha(xy) \quad (x, y \in G)$.

We call (1.6), an automorphism representation of G, with the understanding that a suitable vector space V over K is involved. In order to compute $\alpha(x)$ we refer V to a particular basis, say

$$V = [\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m]. \tag{1.7}$$

Then the action of $\alpha(x)$ is described by a matrix, $A(x) = (a_{ij}(x))$ over K, where

$$\mathbf{p}_i\alpha(x) = \sum_{j=1}^{n} a_{ij}(x)\mathbf{p}_j \quad (i = 1, 2, \ldots, m). \tag{1.8}$$

As $\alpha(x)\alpha(y) = \alpha(xy)$, the matrix function $A(x)$ satisfies $A(x)A(y) = A(xy)$.

When the basis of V is changed, $\alpha(x)$ is described by a matrix of the form $B(x) = T^{-1}A(x)T$, where T is a non-singular matrix over K which is independent of x. Thus a representation $\alpha(x)$ gives rise to a class of equivalent matrix representations $A(x), B(x), \ldots$. Conversely, if we start with

a matrix representation $A(x)$ we can associate with it an automorphism representation $\alpha(x)$ by starting with an arbitrary vector space V given by (1.7) and defining the action of $\alpha(x)$ by means of (1.8). Thus we can say that the classes of equivalent matrix representations are in one-to-one correspondence with automorphism representations of suitable vector spaces.

In an automorphism representation, each element x of G is associated with an automorphism $\alpha(x)$ of V. We denote this automorphism by x ; that is, we put

$$\mathbf{v}x = \mathbf{v}\alpha(x) \tag{1.9}$$

We say that G acts on V in accordance with (1.9). Formally, this defines a right-hand multiplication of a vector in V by an element of G.

Definition 2. *Let G be a group. The vector space V over K is called a G-**module** , if a multiplication $\mathbf{v}x$ $(\mathbf{v} \in V, x \in G)$ is defined, subject to the rules:*

(i) $\mathbf{v}x \in V$;

(ii) $(h\mathbf{v} + k\mathbf{w})x = h(\mathbf{v}x) + k(\mathbf{w}x)$, $(\mathbf{v}, \mathbf{w} \in V; h, k \in K)$;

(iii) $\mathbf{v}(xy) = (\mathbf{v}x)y$;

(iv) $\mathbf{v}1 = \mathbf{v}$

This definition recaptures the notion of an automorphism representation.

(i) states that multiplication by x induces a map of V into itself;
(ii) expresses that this map is linear; (iii) establishes the homomorphic property; finally,
(iv) imply that x and x^{-1} induce mutually inverse maps so that all these maps are invertible.
If V is a G-module, we say that V affords the automorphism representation defined in (1.9) or else the matrix representation A(x) given by (1.8), except that we now write $\mathbf{p_i x}$ instead of $\mathbf{p_i}\alpha(\mathbf{x})$, thus (1.8) become

$$\mathbf{p}_i x = \sum_{j=1}^{n} a_{ij}(x)\mathbf{p}_j \quad (i = 1, 2, \ldots, m). \tag{1.10}$$

1.3 Characters

Let $A(x) = (a_{ij}(x))$ be a matrix representation of G of degree m. Consider the characteristic polynomial of $A(x)$, namely

$$\det(\lambda I - A(x)) = \begin{vmatrix} \lambda - a_{11}(x) & -a_{12}(x) & \ldots & -a_{1m}(x) \\ -a_{21}(x) & \lambda - a_{22}(x) & \ldots & -a_{2m}(x) \\ \ldots & \ldots & \ldots & \ldots \\ -a_{m1}(x) & -a_{m2}(x) & \ldots & \lambda - a_{mm}(x) \end{vmatrix}$$

This is a polynomial of degree m in λ, and the coefficient of $-\lambda^{m-1}$ is $\phi(x) = a_{11}(x) + a_{22}(x) + \cdots + a_{mm}(x)$. That is $\phi(x) = \text{tr } A(x)$.
We regard $\phi(x)$ as a function on G with values in K, and we call it the character of $A(x)$.

Proposition 3. *Let $A(x)$ be a matrix representation of G. Then the character $\phi(x) = tr\ A(x)$ has the following proper-*

ties:
(i) equivalent representations have the same character;
(ii) if x and y are conjugate in G, then $\phi(x) = \phi(y)$.

Proof. (i) If $B(x) = T^{-1}A(x)T$ is a representation equivalent to $A(x)$, then

$$\det(\lambda I - B(x)) = \det(\lambda I - A(x)), \tag{1.11}$$

as $\lambda I - B(x) = \lambda I - T^{-1}A(x)T = T^{-1}(\lambda I - A(x))T$ and hence $\det(\lambda I - B(x)) = \det(T^{-1}) \det(\lambda I - A(x)) \det T$.
On comparing coefficients of λ^{m-1} in (1.11) we get
$b_{11}(x)+b_{22}(x)+\cdots+b_{mm}(x) = a_{11}(x)+a_{22}(x)+\cdots+a_{mm}(x)$, that is, equivalent representations have the same character.
(ii) Suppose that x and $y = t^{-1}xt$ are conjugate elements of G. Then in any matrix representation $A(x)$ we have $A(y) = A(t^{-1}xt) = (A(t))^{-1}A(x)A(t)$. On taking traces on both sides. we get $trA(y) = trA(x)$.(As $A(x)$ and $A(y)$ are similar matrices, they have the same trace.) □

Note. *Thus, in every representation, the character is constant throughout each conjugacy class of G. A function having the same value throughout a conjugacy class is called a class function. So, we say that ϕ is a class function on G.*

1.4 Reducibility

Definition 4. *Let V be a G-module over K. We say that U is a **submodule** of V if*

(i) U is a vector space (over K) contained in V, and
(ii) U is a G-module, that is $ux \in U$ for all $u \in U$ and $x \in G$. Every G-module V possesses the trivial submodules $U = V$ and $U = 0 = \{0\}$.

A non-trivial submodule is also called a proper submodule.

Definition 5. *A G-module is said to be **reducible** over K if it possesses a proper submodule; otherwise it is said to be irreducible over K.*

Now we translate the notion of reducibility into matrix language. Thus suppose that the G-module V of dimension m possesses a submodule U of dimension r, where $0 < r < m$. Let $A(x)$ be the matrix representation

$$\mathbf{p}_i\alpha(x) = \sum_{j=1}^{m} a_{ij}(x)\mathbf{p}_j \quad (i = 1, 2, \ldots, m). \tag{1.12}$$

afforded by V relative to the basis

$$V = [\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m]. \tag{1.13}$$

Now refer V to a new basis, which is adapted to the subspace U. This process consists in first choosing an arbitrary basis for U, say $U = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r]$, and then extending this to a basis of V by suitable vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_s$, where $r + s = m$. Thus V has a basis of the form

$$V = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_s]. \tag{1.14}$$

The relationship between the bases (1.13) and (1.14) is ex-

pressed by a matrix $T = (t_{i\lambda})$ and, when the basis (1.14) is used, the action of x is described by the matrix $B(x) = T^{-1}A(x)T$.

To obtain more information about $B(x)$, we study the action of x by operating directly on the vectors in the basis (1.14). As U is itself a G-module, $\mathbf{u}_i x$ lies in U, and hence we have

$$\mathbf{u}_i x = \sum_{j=1}^{r} c_{ij}(x)\mathbf{u}_j \quad (i = 1, 2, \ldots, r), \tag{1.15}$$

where $c_{ij}(x) \in K$. About $\mathbf{w}_h x$, we can only state that it is a vector in V and so

$$\mathbf{w}_h x = \sum_{j=1}^{r} e_{hj}(x)\mathbf{u}_j + \sum_{k=1}^{s} d_{hk}(x)\mathbf{w}_k \quad (h = 1, 2, \ldots, s), \tag{1.16}$$

where $e_{hj}(x), d_{hk}(x) \in K$. So we get

$$B(x) = \begin{pmatrix} C(x) & 0 \\ E(x) & D(x) \end{pmatrix} \tag{1.17}$$

where $C(x) = (c_{ij})_{r\times r}$, $E(x) = (e_{hj})_{s\times r}$, and $D(x) = (d_{hk})_{s\times s}$. So the matrix analogue of Definition 5 is as follows.

Definition 6. *The matrix representation $A(x)$ is **reducible** over K if there exists a non-singular matrix T over K such that $B(x) = T^{-1}A(x)T = \begin{pmatrix} C(x) & 0 \\ E(x) & D(x) \end{pmatrix}$ for all $x \in G$.*

Since $B(x)$ is a matrix representation, we have that, for all x and y in G, $B(xy) = B(x)B(y)$ and hence

$$\begin{pmatrix} C(xy) & 0 \\ E(xy) & D(xy) \end{pmatrix} = \begin{pmatrix} C(x) & 0 \\ E(x) & D(x) \end{pmatrix} \begin{pmatrix} C(y) & 0 \\ E(y) & D(y) \end{pmatrix}$$
$$= \begin{pmatrix} C(x)C(y) & 0 \\ E(x)C(y) + D(x)E(y) & D(x)D(y) \end{pmatrix}.$$

Thus we have $C(xy) = C(x)C(y)$, $E(xy) = E(x)C(y) + D(x)E(y)$, and $D(xy) = D(x)D(y)$.
So we can see that $C(x)$ and $D(x)$ are themselves matrix representations of G. From expression (1.15) we observe that $C(x)$ is the representation afforded by the G-module U referred to the basis $U = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r]$. To construct a G-module for $D(x)$, consider the quotient space V/U. (a vector space is an additive Abelian group. Hence U is a (normal) subgroup of V.) The elements of V/U are all the cosets $U + \mathbf{v}$, the composition law being

$$(U + \mathbf{v}_1) + (U + \mathbf{v}_2) = U + (\mathbf{v}_1 + \mathbf{v}_2). \tag{1.18}$$

V/U is a vector space over K by virtue of the definition

$$h(U + \mathbf{v}) = U + h\mathbf{v} \quad (h \in K). \tag{1.19}$$

The 'natural' map $\pi : V \to V/U$ assigns to each vector $\mathbf{v}$ the coset $U + \mathbf{v}$, in which it lies, that is $\mathbf{v}\pi = U + \mathbf{v}$.

The equations (1.18) and (1.19) can now be rewritten as $\mathbf{v}_1\pi + \mathbf{v}_2\pi = (\mathbf{v}_1 + \mathbf{v}_2)\pi$, $h(\mathbf{v}\pi) = (h\mathbf{v})\pi$, which imply that π is a homomorphism, that is $(h_1\mathbf{v}_1 + h_2\mathbf{v}_2)\pi = h_1(\mathbf{v}_1\pi) + h_2(\mathbf{v}_2\pi)$, where $h_1, h_2 \in K$. The kernel of π is the space U, which is the zero element of V/U. Hence if π is applied to any

relation between vectors of V, those vectors which lie in U are 'killed'. For example, suppose that $2\mathbf{u}_1 + 4\mathbf{v}_1 = 3\mathbf{u}_2 + \mathbf{v}_2$, where $\mathbf{u}_1, \mathbf{u}_2 \in U$. Then $4\mathbf{v}_1\pi = \mathbf{v}_2\pi$.

Using the basis (1.14) we can readily construct a basis for V/U. We assert that

$$V/U = [\mathbf{w}_1\pi, \mathbf{w}_2\pi, \ldots, \mathbf{w}_s\pi]. \tag{1.20}$$

To see this, an arbitrary element of V/U is of the form $\mathbf{v}\pi$, where $\mathbf{v}$ is a suitable vector of V, say

$\mathbf{v} = \sum_{i=1}^{r} a_i\mathbf{u}_i + \sum_{j=1}^{s} b_j\mathbf{w}_j \quad (a_i, b_j \in K)$. On applying π, $\mathbf{v}\pi = \sum_{j=1}^{s} b_j(\mathbf{w}_j\pi)$.

Hence the vectors (1.20) certainly span V/U.

Now to show that they are linearly independent.

Let $\sum_{j=1}^{s} c_j(\mathbf{w}_j\pi) = 0$.

i.e. $(\sum_{j=1}^{s} c_j\mathbf{w}_j)\pi = 0$.

Then $c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_s\mathbf{w}_s$ would lie in the kernel of π, that is in U, and so $c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_s\mathbf{w}_s = d_1\mathbf{u}_1 + d_2\mathbf{u}_2 + \cdots + d_r\mathbf{u}_r$. But, as (1.14) is a basis of V, all the coefficients in the linear combination on the both sides must be zero. Hence the vectors (1.20) are linearly independent and thus $V/U = [\mathbf{w}_1\pi, \mathbf{w}_2\pi, \ldots, \mathbf{w}_s\pi]$.

We may regard V/U as a G-module if we define the action of G by

$$(U + \mathbf{v})x = U + \mathbf{v}x. \tag{1.21}$$

First of all, we must check that this rule is 'well-defined' (that is, it is independent of the element chosen): the left-hand side of (1.21) remains unchanged if we replace $\mathbf{v}$ by

$\mathbf{u}+\mathbf{v}$, where $\mathbf{u}$ is an arbitrary vector of U.
Then the right-hand side of (1.21) is $U+(\mathbf{u}+\mathbf{v})x = U+\mathbf{u}x+\mathbf{v}x = U+\mathbf{v}x$, because $\mathbf{u}x \in U$.
Now we will show that V/U is indeed a G-module.

1. $(U+\mathbf{v})x = U+\mathbf{v}x$ and so $(U+\mathbf{v})x \in V/U$, as $\mathbf{v}x \in V$, V being a G-module.

2. $[h(U+\mathbf{v})+k(U+\mathbf{w})]x = [(U+h\mathbf{v})+(U+k\mathbf{w})]x = [U+(h\mathbf{v}+k\mathbf{w})]x$
   $= U+(h\mathbf{v}+k\mathbf{w})x = (U+h\mathbf{v}x)+(U+k\mathbf{w}x) = h(U+\mathbf{v}x)+k(U+\mathbf{w}x)$, where $x, y \in G$, $h, k \in K$.

3. $(U+\mathbf{v})(xy) = U+\mathbf{v}(xy) = (U+(\mathbf{v}x))y = ((U+\mathbf{v})x)y$.

4. $(U+\mathbf{v})1 = U+\mathbf{v}1 = U+\mathbf{v}$.

Therefore V/U is a G-module.
(1.21) can be expressed as

$$(v\pi)x = (vx)\pi. \tag{1.22}$$

i.e. π commutes with the action of G. Applying π to (1.16) and using (1.22) we get
$(\mathbf{w}_h x)\pi = (\sum_{j=1}^{r} e_{hj}(x)\mathbf{u}_j)\pi + (\sum_{k=1}^{s} d_{hk}(x)\mathbf{w}_k)\pi$
i.e. $(\mathbf{w}_h\pi)x = \sum_{j=1}^{r} e_{hj}(x)(\mathbf{u}_j\pi) + \sum_{k=1}^{s} d_{hk}(x)(\mathbf{w}_k\pi) = \sum_{k=1}^{s} d_{hk}(x)(\mathbf{w}_k\pi)$.
which shows that the G-module V/U affords the matrix representation $D(x)$ relative to the basis (1.20). Thus $V/U = [\mathbf{w}_1\pi, \mathbf{w}_2\pi, \ldots, \mathbf{w}_s\pi]$.

It may happen that the submodule U is irreducible; if not, it possesses a proper submodule U', which, of course, is also a submodule of V. Continuing in this manner we arrive at an irreducible submodule U_1 of V. Alternatively, we can say that, if V is reducible, it must possess a proper submodule of minimal dimension; such a minimal submodule is then necessarily irreducible. Thus, without loss of generality, we may assume that, if $A(x)$ is reducible, there exists a non-singular matrix T such that $T^{-1}A(x)T = \begin{pmatrix} A_1(x) & 0 \\ E(x) & D(x) \end{pmatrix}$, where $A_1(x)$ is irreducible. If now $D(x)$ is reducible, the process can be continued until we reach the situation described in the following theorem:

Theorem 7. *Let $A(x)$ be a matrix representation of G of degree m over K. Then either $A(x)$ is irreducible or else*

$$A(x) \sim \begin{pmatrix} A_1(x) & 0 & 0 & \dots & 0 \\ A_{21}(x) & A_2(x) & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ A_{l1}(x) & A_{l2}(x) & A_{l3}(x) & \dots & A_l(x) \end{pmatrix} \tag{1.23}$$

where $A_1(x)$, $A_2(x)$, ..., $A_l(x)$ are irreducible over K.

Proof. We use induction with respect to m. When $m = 1$, $A(x)$ is certainly irreducible. Suppose now that $m > 1$ and that $A(x)$ is reducible. Hence there exists a non-singular matrix T such that $T^{-1}A(x)T = \begin{pmatrix} A_1(x) & 0 \\ E(x) & D(x) \end{pmatrix}$, where $A_1(x)$ is irreducible. If $D(x)$ is irreducible, we have estab-

lished the result with $l = 2$. Otherwise, let $\dim A_1(x) = r$, $\dim D(x) = s$.

By induction we assume that there exists a non-singular matrix S_0 over K of degree s such that

$$S_0^{-1}D(x)S_0 = \begin{pmatrix} A_2(x) & 0 & \dots & 0 \\ A_{32}(x) & A_3(x) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ A_{l2}(x) & A_{l3}(x) & \dots & A_l(x) \end{pmatrix} \tag{1.24}$$

where $A_1(x)$, $A_2(x)$, $\dots$, $A_l(x)$ are irreducible.

Take $S = \begin{pmatrix} I_r & 0 \\ 0 & S_0 \end{pmatrix}$. Then,

$$\begin{aligned} S^{-1}T^{-1}A(x)TS &= \begin{pmatrix} I_r & 0 \\ 0 & S_0^{-1} \end{pmatrix} \begin{pmatrix} A_1(x) & 0 \\ E(x) & D(x) \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & S_0 \end{pmatrix} \\ &= \begin{pmatrix} I_r & 0 \\ 0 & S_0^{-1} \end{pmatrix} \begin{pmatrix} A_1(x) & 0 \\ E(x) & D(x)S_0 \end{pmatrix} \\ &= \begin{pmatrix} A_1(x) & 0 \\ S_0^{-1}E(x) & S_0^{-1}D(x)S_0 \end{pmatrix}. \end{aligned}$$

The proof is concluded by substituting (1.24) in this matrix and partitioning $S_0^{-1}E(x) = \begin{pmatrix} A_{21}(x) \\ A_{31}(x) \\ \vdots \\ A_{l1}(x) \end{pmatrix}$ to conform with the grouping of rows in (1.24). □

Remark 8. *Briefly, this theorem states that every matrix*

representation can be brought into lower triangular block form, in which the diagonal blocks are irreducible.

Together with any representation $A(x)$ we have the contragredient representation $A^\dagger(x) = A'(x^{-1})$. ($A^\dagger(x)$ is to be read as A dagger(x)).

$$\begin{aligned} A^\dagger(x)A^\dagger(y) &= A'(x^{-1})A'(y^{-1}) &&= [A'(x)]^{-1}[A'(y)]^{-1} \\ &= [A'(x)A'(y)]^{-1} &&= [(A(x)A(y))']^{-1} \\ &= [(A(xy))']^{-1} &&= [(A(xy))^{-1}]' \\ &= [(A(xy)^{-1})]' &&= [A((xy)^{-1})]' \\ &= A'[(xy)^{-1}] &&= A^\dagger(xy) \end{aligned}$$

Clearly, $A^{\dagger\dagger}(x) = A(x)$.

If $\phi(x)$ and $\phi^\dagger(x)$ are the characters of $A(x)$ and $A^\dagger(x)$ respectively, then $\phi^\dagger(x) = \overline{\phi(x)}$ since $\phi(x^{-1}) = \overline{\phi(x)}$, and A and A' have the same character. Therefore A and $A^\dagger$ are equivalent if and only if ϕ is real valued. In particular, if χ is a simple character, so is $\overline{\chi}$. (χ is to be read as chi.)

$$\begin{aligned} \langle \overline{\chi}, \overline{\chi} \rangle &= \frac{1}{g} \sum_{x \in G} \overline{\chi}(x)\overline{\chi}(x^{-1}) \\ &= \frac{1}{g} \sum_{x \in G} \chi(x^{-1})\chi(x) \\ &= \langle \chi, \chi \rangle = 1. \quad \text{(Since } \chi \text{ is a simple character)} \end{aligned}$$

1.5 Permutation Representations

Let G be a permutation group of degree m, that is a subgroup of S_m, possibly the whole of S_m, and let

$$x = \begin{pmatrix} 1 & 2 & \dots & m \\ \xi_1 & \xi_2 & \dots & \xi_m \end{pmatrix} \tag{1.25}$$

be a typical element of G. (The symbol ξ is to be read as xi.)

We construct a G-module V as follows:

Starting from an arbitrary vector space $V = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m]$ of dimension m over K, we define the action of G on V by the rule

$$\mathbf{v}_i x = \mathbf{v}_{\xi_i} \quad (i = 1, 2, ..., m). \tag{1.26}$$

Let $y = \begin{pmatrix} 1 & 2 & \dots & m \\ \eta_1 & \eta_2 & \dots & \eta_m \end{pmatrix} = \begin{pmatrix} \xi_1 & \xi_2 & \dots & \xi_m \\ \tau_1 & \tau_2 & \dots & \tau_m \end{pmatrix}$ is another element of G, then $(\mathbf{v}_i x)y = \mathbf{v}_{\xi_i} y = \mathbf{v}_{\tau_i}$.

(The symbol η is to be read as eta and the symbol τ is to be read as tau.)

But we have $xy = \begin{pmatrix} 1 & 2 & \dots & m \\ \xi_1 & \xi_2 & \dots & xi_m \end{pmatrix} \begin{pmatrix} \xi_1 & \xi_2 & \dots & \xi_m \\ \tau_1 & \tau_2 & \dots & \tau_m \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & \dots & m \\ \tau_1 & \tau_2 & \dots & \tau_m \end{pmatrix}$ and therefore $\mathbf{v}_i(xy) = \mathbf{v}_{\tau_i}$.

Thus $\mathbf{v}_i(xy) = (\mathbf{v}_i x)y$ for $i = 1, 2, \dots, m$.

Let $\mathbf{v} = \sum_{i=1}^{m} c_i \mathbf{v}_i \in V$, $x \in G$.

Thus the first axiom for a G-module is satisfied.

Let $\mathbf{w} = \sum_{i=1}^{m} d_i \mathbf{v}_i$.

For $h, k \in K$,

$$
\begin{aligned}
(h\mathbf{v}+k\mathbf{w})x &= (h\sum_{i=1}^{m} c_i\mathbf{v}_i + k\sum_{i=1}^{m} d_i\mathbf{v}_i)x \\
&= (\sum_{i=1}^{m}(hc_i+kd_i)\mathbf{v}_i)x \\
&= \sum_{i=1}^{m}(hc_i+kd_i)(\mathbf{v}_i x) \\
&= \sum_{i=1}^{m}(hc_i+kd_i)\mathbf{v}_{\xi_i} \\
&= h\sum_{i=1}^{m} c_i\mathbf{v}_{\xi_i} + k\sum_{i=1}^{m} d_i\mathbf{v}_{\xi_i} \\
&= h\sum_{i=1}^{m} c_i\mathbf{v}_i x + k\sum_{i=1}^{m} d_i\mathbf{v}_i x \\
&= h(\sum_{i=1}^{m} c_i\mathbf{v}_i)x + k(\sum_{i=1}^{m} d_i\mathbf{v}_i)x \\
&= h(\mathbf{v}x) + k(\mathbf{w}x)
\end{aligned}
$$

This is the second axiom.
$\mathbf{v}x = (\sum_{i=1}^{m} c_i\mathbf{v}_i)x = \sum_{i=1}^{m} c_i(\mathbf{v}_i x)$. Since $\mathbf{v}_i x = \mathbf{v}_{\xi_i} \in V$, and as V is a vector space, $\sum_{i=1}^{m} c_i(\mathbf{v}_i x) \in V$. That is, $\mathbf{v}x \in V$.

$$\begin{aligned}\mathbf{v}(xy) &= (\sum_{i=1}^{m} c_i\mathbf{v}_i)(xy)\\ &= \sum_{i=1}^{m} c_i(\mathbf{v}_i(xy))\\ &= \sum_{i=1}^{m} c_i(\mathbf{v}_i x)y\\ &= (\sum_{i=1}^{m} c_i(\mathbf{v}_i x))y\\ &= (\sum_{i=1}^{m} c_i(\mathbf{v}_i)x)y\\ &= (\mathbf{v}x)y\end{aligned}$$

Thus axiom (3) holds.

$\mathbf{v}1 = (\sum_{i=1}^{m} c_i\mathbf{v}_i)1 = \sum_{i=1}^{m} c_i(\mathbf{v}_i 1) = \sum_{i=1}^{m} c_i\mathbf{v}_i 1 = \mathbf{v}$, which proves axiom (4).

Therefore V is a G-module by virtue of the rule (1.26). For a matrix representation $A(x)$, $\mathbf{p}_i x = \sum_{j=1}^{m} a_{ij}(x)\mathbf{p}_j$ $(i = 1, 2, \ldots, m)$.

Here $\mathbf{v}_i x = \mathbf{v}_{\xi_i}$ $(i = 1, 2, \ldots, m)$. Hence V affords the representation $N(x) = (\delta_{\xi_i j})$. Each $N(x)$ is a permutation matrix; that is, it has exactly one unit in each row and in each column, all the remaining coefficients are zero. The representation is called the natural representation of G. Thus every permutation group possesses a natural representation.

To find the natural character $\nu(x)$ of G, that is the character of $N(x)$ ($\nu(x)$ is to be read as nu(x))

$\nu(x) = trN(x) = \sum_{i=1}^{m} \delta_{\xi_i i}$

Now $\delta_{\xi_i i}$ is zero unless $i = \xi_i$, in which case it is equal to unity. But $i = \xi_i$ means that the object i remains fixed by x. Hence $\nu(x)$ =number of objects fixed by x.

Example 9. *To determine the natural representation of the symmetric group S_3. The six elements of S_3 are* $1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, $a = (12) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$, $b = (13) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$, $c = (23) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, $e = (123) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $f = (132) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$.

The action of S_3 on the vector space

$$V = [v_1, v_2, v_3] \tag{1.27}$$

is explicitly given as follows:

$(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)1 = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$

$(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)a = (\mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_3)$

$(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)b = (\mathbf{v}_3, \mathbf{v}_2, \mathbf{v}_1)$

$(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)c = (\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_2)$

$(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)e = (\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_1)$

$(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)f = (\mathbf{v}_3, \mathbf{v}_1, \mathbf{v}_2)$

In order to obtain the corresponding matrices $N(x)$, we have to express the right-hand side of each of these equations linearly in terms of the basis elements (1.27) in their original order, and to find the coefficients that arise. Thus

$$N(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; N(a) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$N(b) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; N(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$N(e) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \; N(f) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Therefore the characters are given by $\nu(1) = 3, \nu(a) = \nu(b) = \nu(c) = 1, \nu(e) = \nu(f) = 0$ *(trace of the corresponding matrix).*

When $m \geq 2$, the natural representation is always reducible. Since each x merely permutes the basis vectors of V, the vector $\mathbf{u} = \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_m$ has the property that

$$\mathbf{u}x = \mathbf{u} \quad (x \in G). \tag{1.28}$$

Hence the one-dimensional space $U = [u]$ is a submodule of V. (1.28) implies that U affords the trivial representation (since $\alpha(x) = 1, \forall x \in G$). If we change the basis of V so as to include u as one of the basis vectors, then

$$N(x) \sim \begin{bmatrix} 1 & 0 \\ E(x) & D(x) \end{bmatrix} = M(x) \tag{1.29}$$

say, where $D(x)$ is a representation of degree $m-1$ and $E(x)$ is an $(m-1) \times 1$ matrix.

Example 10. *Consider this construction in the case of* S_3.

We choose the basis $V = [\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2]$, *where* $u = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$. *The vector* $\mathbf{v}_3$ *is to be replaced by* $\mathbf{u} - \mathbf{v}_1 - \mathbf{v}_2$. *Relative to the new basis the action of* G *is as follows:*

$(\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2)1 = (\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2)$

$(\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2)a = (\mathbf{u}a, \mathbf{v}_1a, \mathbf{v}_2a) = (\mathbf{u}, \mathbf{v}_2, \mathbf{v}_1)$

$(\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2)b = (\mathbf{u}b, \mathbf{v}_1b, \mathbf{v}_2b) = (\mathbf{u}, \mathbf{v}_3, \mathbf{v}_2) = (\mathbf{u}, \mathbf{u} - \mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_2)$

$(\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2)c = (\mathbf{u}c, \mathbf{v}_1c, \mathbf{v}_2c) = (\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2) = (\mathbf{u}, \mathbf{v}_1, \mathbf{u} - \mathbf{v}_1 - \mathbf{v}_2)$

$(\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2)e = (\mathbf{u}e, \mathbf{v}_1e, \mathbf{v}_2e) = (\mathbf{u}, \mathbf{v}_2, \mathbf{v}_3) = (\mathbf{u}, \mathbf{v}_2, \mathbf{u} - \mathbf{v}_1 - \mathbf{v}_2)$

$(\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2)f = (\mathbf{u}f, \mathbf{v}_1f, \mathbf{v}_2f) = (\mathbf{u}, \mathbf{v}_3, \mathbf{v}_2) = (\mathbf{u}, \mathbf{u} - \mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_1)$

Expressing this information in matrix language, we find the representation $M(x)$, *which is equivalent to* $N(x)$;

$$M(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ M(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$M(b) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix},\ M(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix},$$

$$M(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix},\ M(f) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The list of the matrices $D(x)$, *which furnish a two-dimensional representation of* S_3 *is*

$$D(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ D(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ D(b) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix},$$

$$D(c) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix},\ D(e) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},\ D(f) =$$

$$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

1.6 Complete Reducibility

A reducible matrix representation $A(x)$ can be brought into the triangular shape $B(x) = \begin{pmatrix} C(x) & 0 \\ E(x) & D(x) \end{pmatrix}$. If we could remove the off-diagonal block $E(x)$ by a further transformation, we would get $A(x) \sim \begin{pmatrix} C(x) & 0 \\ 0 & D(x) \end{pmatrix}$ or $A(x) \sim diag(C(x), D(x))$ in concise notation. This aim cannot always be achieved, as is shown by the following example:

Let $G : \ldots, x^{-2}, x^{-1}, x^0(= 1), x^1, x^2, \ldots$ be the infinite cyclic group generated by x. Then $A(x^h) = \begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix}$ $(h = \ldots, -2, -1, 0, 1, 2, \ldots)$ is a representation of G, because

$$A(x^h)A(x^k) = \begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ h+k & 1 \end{pmatrix} =$$

$A(x^{h+k}) = A(x^h x^k)$.

But when $h \neq 0$, it is impossible to transform $A(x^h)$ into diagonal form; for both latent roots of $A(x^h)$ are equal to unity so that the diagonal form would have to be $diag(1, 1) = I$, which gives $A(x^h) = I$ for all h, which is not true.

Theorem 11 (Maschke's Theorem)**.** *Let G be a finite group of order g, and let K be a field whose characteristic is ei-*

ther equal to zero or else is prime to g. Suppose that $A(x)$ is a matrix representation of G over K such that $A(x) \sim \begin{pmatrix} C(x) & 0 \\ E(x) & D(x) \end{pmatrix}$. Then $A(x) \sim diag(C(x), D(x))$.

Remark 12. *The condition 'characteristic is either equal to zero or else is prime to g' imposed on the field K is required for the existence of g^{-1} in K.*

Proof. Since $A(x)$ may be replaced by an equivalent representation, with out loss of generality, we assume that $A(x) = \begin{pmatrix} C(x) & 0 \\ E(x) & D(x) \end{pmatrix}$, where $C(x)$ and $D(x)$ are square matrices of degrees r and s respectively. We seek a non-singular matrix T over K which is independent of x and has the property that

$$T^{-1}A(x)T = \begin{pmatrix} C(x) & 0 \\ 0 & D(x) \end{pmatrix} \tag{1.30}$$

Putting $T = \begin{pmatrix} I_r & 0 \\ Q & I_s \end{pmatrix}$, $T^{-1} = \begin{pmatrix} I_r & 0 \\ -Q & I_s \end{pmatrix}$, where Q is an $s \times r$ matrix to be determined.
Thus (1.30) become

$$\begin{pmatrix} I_r & 0 \\ -Q & I_s \end{pmatrix} \begin{pmatrix} C(x) & 0 \\ E(x) & D(x) \end{pmatrix} \begin{pmatrix} I_r & 0 \\ Q & I_s \end{pmatrix} = \begin{pmatrix} C(x) & 0 \\ 0 & D(x) \end{pmatrix}.$$

Multiplying both sides by T, $\begin{pmatrix} C(x) & 0 \\ E(x) & D(x) \end{pmatrix} \begin{pmatrix} I_r & 0 \\ Q & I_s \end{pmatrix} =$

$\begin{pmatrix} I_r & 0 \\ Q & I_s \end{pmatrix} \begin{pmatrix} C(x) & 0 \\ 0 & D(x) \end{pmatrix}$. On expanding the matrix products and comparing corresponding blocks on both sides we find that $E(x) + D(x)Q = QC(x)$ is the sole condition for Q. This has to hold simultaneously for all $x \in G$.
We have $C(xy) = C(x)C(y)$
$D(xy) = D(x)D(y)$, and

$$E(xy) = E(x)C(y) + D(x)E(y). \tag{1.31}$$

Multiplying (1.31) throughout by $C(y^{-1})$, we get $E(xy)C(y^{-1}) = E(x) + D(x)E(y)C(y^{-1})$. We regard x as an arbitrary but fixed element of G, while y runs over G. Summing over y,

$$\sum_{y\in G} E(xy)C(y^{-1}) = gE(x) + D(x)\sum_{y\in G} E(y)C(y^{-1}). \tag{1.32}$$

On the left put $z = xy$. So when y runs over G so does z. Hence the left-hand side is $\sum_{z\in G} E(z)C(z^{-1}x) = \sum_{z\in G} E(z)C(z^{-1})C(x) = \sum_{y\in G} E(y)C(y^{-1})C(x)$.
Define $Q = \sum_{y\in G} E(y)C(y^{-1})$.
Dividing (1.32) by g, we get $QC(x) = E(x) + D(x)Q$, which is the condition to be satisfied by the matrix Q. □

Now we formulate this result in the language of module theory. Suppose the G-module V affords the reducible representation $A(x)$. Then V possesses a proper submodule, $U = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r]$, which affords the representation $C(x)$ in $\begin{pmatrix} C(x) & 0 \\ E(x) & D(x) \end{pmatrix}$.

Maschke's Theorem asserts that there exists a basis $V = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_s]$ relative to which all the coefficients $e_{hj}(x)$ in the expression,
$\mathbf{w}_h x = \sum_{j=1}^{r} e_{hj}(x)\mathbf{u}_j + \sum_{k=1}^{s} d_{hk}(x)\mathbf{w}_k \quad (h = 1, 2, \ldots, s)$ are equal to zero so that
$\mathbf{w}_h x = \sum_{k=1}^{s} d_{hk}(x)\mathbf{w}_k \quad (h = 1, 2, \ldots, s).$ This means that $W = [\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_s]$ is a G-module which affords the representation $D(x)$.
Then we write $V = U \oplus W$ and say that V is the direct sum of the G-modules U and W.
Maschke's Theorem can be applied at each stage of the process that leads to Theorem 7; thus all non-diagonal blocks in the lower triangular block form stated in the theorem can be reduced to zero by a suitable equivalence transformation. Thus we get the following definition.

Definition 13. *A matrix representation $A(x)$ over K is said to be* ***completely reducible*** *if $A(x) \sim diag(A_1(x), A_2(x), \ldots, A_l(x))$, where $A_i(x)$ $(i = 1, 2, \ldots, l)$ are irreducible representations over K. Again, the G-module V over K is said to be completely reducible, if $V = U_1 \oplus U_2 \oplus \cdots \oplus U_l$, where U_i $(i = 1, 2, \ldots, l)$ are irreducible G-modules over K.*

Note. *The case $l = 1$ refers to an irreducible representation $A(x)$.*

Thus we have the more general version of Maschke's The-

orem:

Theorem 14. *Let G be a finite group of order g, and let K be a field whose characteristic is zero or else prime to g. Then*
(i) every matrix representation of G over K is completely reducible, or equivalently,
(ii) every G-module over K is completely reducible.

Note. *In the remaining part, we deal with finite groups and with ground fields of characteristic zero. We shall therefore assume from now on that all representations are completely reducible.*

1.7 Schur's Lemma

We first consider the original matrix version used by Schur.

Theorem 15. *Let $A(x)$ and $B(x)$ be two irreducible representations over K of a group G, and suppose there exists a constant matrix T over K such that $TA(x) = B(x)T$ for all x in G. Then*
(i) either $T = 0$ or
(ii) T is non-singular so that $A(x) = T^{-1}B(x)T$, that is $A(x)$ and $B(x)$ are equivalent.

We prefer to translate Schur's lemma into the language of G-modules, as this will provide a better understanding of the situation.

Definition 16. *Let V and U be G-modules over K, not necessarily irreducible. The linear map $\theta : V \to U$ is called a G-**homomorphism** if it commutes with all the linear maps induced by the action of G, that is if*

$$(vx)\theta = (v\theta)x \tag{1.33}$$

for all $v \in V$ and $x \in G$.

Suppose now that U and V afford the matrix representations $A(x)$ and $B(x)$ relative to the bases $U = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m]$ and $V = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$ respectively. Thus $\mathbf{u}_i x = \sum_{j=1}^{m} a_{ij}(x)\mathbf{u}_j$ $(i = 1, 2, \ldots, m)$, and $\mathbf{v}_h x = \sum_{k=1}^{n} b_{hk}(x)\mathbf{v}_k$ $(h = 1, 2, \ldots, n)$.
The linear map θ is described by an $n \times m$ matrix $T = (t_{hi})$ whose coefficients appear in the equations

$$v_h\theta = \sum_{i=1}^{m} t_{hi}\mathbf{u}_i, \quad h = 1, 2, \ldots, n \tag{1.34}$$

From equation (1.33) applied to a typical basis vector v_h, we get

$$\begin{aligned}(v_h x)\theta &= \left(\sum_{k=1}^{n} b_{hk}(x)\mathbf{v}_k\right)\theta \\ &= \sum_{k=1}^{n} b_{hk}(x)\mathbf{v}_k\theta \\ &= \sum_{k=1}^{n} b_{hk}(x)\sum_{j=1}^{m} t_{kj}\mathbf{u}_j \\ &= \sum_{j=1}^{m}\left(\sum_{k=1}^{n} b_{hk}(x)t_{kj}\right)u_j\end{aligned}$$

$$
\begin{aligned}
(v_h\theta)x &= \left(\sum_{i=1}^{m} t_{hi}\mathbf{u}_i\right)x \\
&= \sum_{i=1}^{m} t_{hi}(\mathbf{u}_i x) \\
&= \sum_{i=1}^{m} t_{hi} \sum_{j=1}^{m} a_{ij}(x)\mathbf{u}_j \\
&= \sum_{j=1}^{m} (\sum_{i=1}^{m} t_{hi}a_{ij}(x))u_j
\end{aligned}
$$

Hence the condition (1.33) for a G-homomorphism θ reduces to $\sum_{i=1}^{m} t_{hi}a_{ij}(x) = \sum_{k=1}^{n} b_{hk}(x)t_{kj} \quad (h = 1, 2, \ldots, n)$. This is the matrix equation $TA(x) = B(x)T \quad (x \in G)$.

Conversely, if T satisfies the relation $TA(x) = B(x)T \quad (x \in G)$, we can define θ by (1.34) and by linearity, we obtain that the condition (1.33) for θ being a G-homomorphism is satisfied. Thus the two conditions are essentially equivalent, and we denote the correspondence between θ and T by $\theta \leftrightarrow T$, assuming that definite bases have been chosen for V and U.

We know that if $\theta : V \to U$ is a linear map, then the kernel of θ, that is $ker\ \theta = \{\mathbf{v}_0 \in V : \mathbf{v}_0\theta = \mathbf{0}\}$, is a subspace of V, and the image of θ, that is $im\ \theta = \{\mathbf{u}_1 \in U : \mathbf{u}_1 = \mathbf{v}_1\theta \text{ for some } \mathbf{v}_1 \in V\}$, is a subspace of U. The map θ is called injective if $ker\ \theta = \{\mathbf{0}\}$, and it is said to be surjective if $im\ \theta = U$. If θ is both injective and surjective, it is termed bijective or an isomorphism; in that case, the matrix T as-

sociated with θ is non-singular.

These concepts apply to G-homomorphisms also. Here we have the important additional property that both $ker\ \theta$ and $im\ \theta$ are themselves G-modules. To see this, first let $\mathbf{v}_0 \in ker\ \theta$. Then $\mathbf{v}_0\theta = \mathbf{0}$; thus $(\mathbf{v}_0 x)\theta = (\mathbf{v}_0)\theta x = 0$. Thus $\mathbf{v}_0 x \in ker\ \theta$ which proves that $ker\ \theta$ is a G-module. Similarly, if $\mathbf{u}_1 \in im\ \theta$, then $\mathbf{u}_1 = \mathbf{v}_1\theta$, and hence it follows that $\mathbf{u}_1 x = (\mathbf{v}_1\theta)x = (\mathbf{v}_1 x)\theta$, and so $\mathbf{u}_1 x \in im\ \theta$ (as $\mathbf{v}_1 x \in V$, V being a G-module). This shows that $im\ \theta$ is a G-module.

So now we consider the abstract version of Schur's Lemma.

Theorem 17. *Suppose that U and V are irreducible G-modules over K. Then a G-homomorphism $\theta : V \to U$ is*
(i) either the zero map, or else
(ii) θ is an isomorphism.

Proof. It is sufficient to prove that if $\theta \neq 0$, then θ is a bijection, that is $ker\ \theta = \{\mathbf{0}\}$ and $im\ \theta = U$. Since $ker\ \theta$ is a submodule of the irreducible G-module V, we must have that either $ker\ \theta = \{\mathbf{0}\}$ or $ker\ \theta = V$. But $ker\ \theta = V$ gives $\theta = 0$; hence we get $ker\ \theta = \{0\}$. Next, since $im\ \theta$ is a submodule of U, either $im\ \theta = \{\mathbf{0}\}$ or $im\ \theta = U$; If $im\ \theta = \{\mathbf{0}\}$, we get $\theta = 0$ and hence it follows that $im\ \theta = U$. So θ is an isomorphism and this proves Schur's Lemma. □

Corollary 18. *Let $A(x)$ be an irreducible matrix representation of G over an algebraically closed field. Then the only matrices which commute with all the matrices $A(x)$ $(x \in G)$ are the scalar multiples of the unit matrix.*

Proof. Let $A(x) = B(x)$ in Schur's Lemma. Given $A(x)$ is irreducible over the algebraically closed field K. Let T be a square matrix over K which satisfies $TA(x) = A(x)T$ for all $x \in G$. Then if k is any 'scalar', we have

$$(kI - T)A(x) = A(x)(kI - T) \quad (x \in G). \tag{1.35}$$

Consider the polynomial equation $\det(kI - T) = 0$. Since K is algebraically closed, there exists a scalar k_0 such that $\det(k_0 I - T) = 0$. The matrix $(k_0 I - T)$ is therefore singular. Applying Schur's Lemma to (1.35) with $k = k_0$, the only possibility is that $(k_0 I - T) = 0$.
That is, $T = k_0 I$.
Conversely, it is obvious that every matrix of the form kI $(k \in K)$ commutes with all the $A(x)$. □

1.8 The commutant (endomorphism) algebra

Here we address the question whether the fact that only the scalar multiples of the unit matrix commute with $A(x)$ implies that $A(x)$ is irreducible.
This is part of a more general problem.
Suppose that $A(x)$ is a completely reducible matrix representation of G over the algebraically closed field K, and let $\mathscr{C}(A)$ be the set of all matrices T over K such that

$$TA(x) = A(x)T \text{ for all } x \in G \tag{1.36}$$

We can see that, if T_1 and T_2 satisfy (1.36), so do the matrices $k_1T_1+k_2T_2$ $(k_1, k_2 \in K)$ and T_1T_2. Hence $\mathscr{C}(A)$ is an algebra over K, that is a vector space endowed with (associative) multiplication; in fact, it is a subalgebra of the complete matrix algebra $M_m(K)$, where m is the degree of A. We call $\mathscr{C}(A)$ the commutant algebra of A.

Proposition 19. *If A and B are equivalent representations, then*
$\mathscr{C}(A) \cong \mathscr{C}(B)$.

Proof. Let $B(x) = P^{-1}A(x)P$.
$T \in \mathscr{C}(A) \implies TA(x) = A(x)T$.
Then $(P^{-1}TP)B(x) = P^{-1}TPP^{-1}A(x)P = P^{-1}TA(x)P$.
$B(x)(P^{-1}TP) = P^{-1}A(x)PP^{-1}TP = P^{-1}A(x)TP = P^{-1}TA(x)P$.
Thus $P^{-1}TP \in \mathscr{C}(B)$.
Conversely, let $P^{-1}TP \in \mathscr{C}(B)$.
Then $(P^{-1}TP)B(x) = B(x)(P^{-1}TP)$
$\implies (P^{-1}TP)P^{-1}A(x)P = P^{-1}A(x)P(P^{-1}TP)$
$\implies P^{-1}TA(x)P = P^{-1}A(x)TP \implies TA(x) = A(x)T$.
Thus $T \in \mathscr{C}(A)$.
Also, the map $\psi : T \to P^{-1}TP$ establishes an algebra-isomorphism between $\mathscr{C}(A)$ and $\mathscr{C}(B)$, because
$(k_1T_1 + k_2T_2)\psi = P^{-1}(k_1T_1 + k_2T_2)P$
$= k_1P^{-1}T_1P + k_2P^{-1}T_2P = k_1T_1\psi + k_2T_2\psi$, and
$(T_1T_2)\psi = P^{-1}(T_1T_2)P = (P^{-1}T_1P)(P^{-1}T_2P) = (T_1\psi)(T_2\psi)$.

Hence $\mathscr{C}(A) \cong \mathscr{C}(B)$. □

Consider the commutant algebra of A, $\mathscr{C}(A)$. We shall be concerned only with properties that are common to isomorphic algebras. So we assume that

$$A(x) = diag(A_1(x), \ldots, A_s(x)) \tag{1.37}$$

where the representations $A_1, A_2, \ldots, A_s$ are irreducible over K, though not necessarily inequivalent to one another.
Let m_i = degree of A_i ; $(i = 1, 2, \ldots, s)$.
$\mathscr{C}(A)$ be the set of all matrices T over K such that condition (1.36) is satisfied. We partition T into blocks, thus

$T = \begin{pmatrix} T_{11} & T_{12} & \ldots & T_{1s} \\ T_{21} & T_{22} & \ldots & T_{2s} \\ \ldots & \ldots & \ldots & \ldots \\ T_{s1} & T_{s2} & \ldots & T_{ss} \end{pmatrix}$ where T_{ij} is an $m_i \times m_j$ matrix.

Substituting for $A(x)$ from (1.37), condition (1.36) become

$$\begin{pmatrix} T_{11} & T_{12} & \ldots & T_{1s} \\ T_{21} & T_{22} & \ldots & T_{2s} \\ \ldots & \ldots & \ldots & \ldots \\ T_{s1} & T_{s2} & \ldots & T_{ss} \end{pmatrix} \begin{pmatrix} A_1(x) & 0 & \ldots & 0 \\ 0 & A_2(x) & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & A_s(x) \end{pmatrix} =$$

$$\begin{pmatrix} A_1(x) & 0 & \ldots & 0 \\ 0 & A_2(x) & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & A_s(x) \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} & \ldots & T_{1s} \\ T_{21} & T_{22} & \ldots & T_{2s} \\ \ldots & \ldots & \ldots & \ldots \\ T_{s1} & T_{s2} & \ldots & T_{ss} \end{pmatrix}$$

$$\Longrightarrow \quad \begin{pmatrix} T_{11}A_1(x) & T_{12}A_2(x) & \dots & T_{1s}A_s(x) \\ T_{21}A_1(x) & T_{22}A_2(x) & \dots & T_{2s}A_s(x) \\ \dots & \dots & \dots & \dots \\ T_{s1}A_1(x) & T_{s2}A_2(x) & \dots & T_{ss}A_s(x) \end{pmatrix} =$$

$$\begin{pmatrix} A_1(x)T_{11} & A_1(x)T_{12} & \dots & A_1(x)T_{1s} \\ A_2(x)T_{21} & A_2(x)T_{22} & \dots & A_2(x)T_{2s} \\ \dots & \dots & \dots & \dots \\ A_s(x)T_{s1} & A_s(x)T_{s2} & \dots & A_s(x)T_{ss} \end{pmatrix}.$$

Thus we get the s^2 equations $T_{ij}A_j(x) = A_i(x)T_{ij}$ $(i, j = 1, 2, \dots, s)$ for the matrix blocks.

To obtain T_{ij}, we apply Schur's Lemma, together with the corollary. Thus

$$T_{ij} = \begin{cases} x_i I_{m_i}, & A_i \sim A_j \\ 0, & A_i \nsim A_j \end{cases} \tag{1.38}$$

where x_i is an arbitrary element of K and I_{m_i} is the unit matrix of degree m_i. Conversely, any matrix of the form (1.38) satisfies (1.36).

Now we obtain a more precise description of $\mathscr{C}(A)$, taking account of the exact number of mutually inequivalent representations in (1.37) and of the multiplicities with which they occur.

Thus without loss of generality, it may be assumed that

$$A(x) = diag(\underbrace{F_1(x), \dots, F_1(x)}_{e_1 times}, \underbrace{F_2(x), \dots, F_2(x)}_{e_2 times}, \underbrace{F_l(x), \dots, F_l(x)}_{e_l times}), \tag{1.39}$$

where $F_1, F_2, \dots, F_l$ are inequivalent irreducible representations of degrees $f_1, f_2, \dots, f_l$, which occur with multiplicities

$e_1, e_2, \ldots, e_l$ in the diagonal form of A. We introduce the Kronecker product (tensor or direct product) of matrices. We are here concerned only with square matrices: let $P = (p_{ij})$ and Q be matrices of degrees m and n respectively. Then we define the $mn \times mn$ matrix

$$P \otimes Q = \begin{pmatrix} p_{11}Q & p_{12}Q & \ldots p_{1m}Q \\ p_{21}Q & p_{22}Q & \ldots p_{2m}Q \\ \ldots & \ldots & \ldots\ldots \\ p_{m1}Q & p_{m2}Q & \ldots p_{mm}Q \end{pmatrix} = (p_{ij}Q).$$

Similarly, if R is of degree m and S is of degree n, then $R \otimes S = (r_{ij}S)$.

$$(P \otimes Q)(R \otimes S) = \begin{pmatrix} \sum_{j=1}^{m} p_{1j}r_{j1}QS & \ldots & \sum_{j=1}^{m} p_{1j}r_{jm}QS \\ \ldots & \ldots & \ldots \\ \sum_{j=1}^{m} p_{mj}r_{j1}QS & \ldots & \sum_{j=1}^{m} p_{mj}r_{jm}QS \end{pmatrix}.$$

So the (i,j)th block in the product is $(\sum_{k=1}^{m} p_{ik}r_{kj})QS$.

Hence we have obtained that $(P \otimes Q)(R \otimes S) = PR \otimes QS$.

We have $I \otimes Q = diag(Q, Q, \ldots, Q)$.

Now consider the direct sum of matrices. We say that the matrix $A = diag(A_1, A_2, \ldots, A_s)$ is the direct sum of the matrices $A_1, A_2, \ldots, A_s$, and write $A = A_1 \oplus A_2 \oplus \cdots \oplus A_s = \sum_{i=1}^{s} \oplus A_i$.

Consider the commutant algebra $\mathscr{C}(A)$.

A representation $D(x) = diag(F(x), \ldots, F(x))$ can be written as $I_e \otimes F(x) = D(x)$, if $F(x)$ is an irreducible representation of degree f is repeated e times.

Theorem 20. *Let $A(x)$ be a completely reducible represen-*

tation of a group G over an algebraically closed field K, and suppose that
$A(x) = \sum_{i=1}^{l} \oplus (I_{e_i} \otimes F_i(x))$ *where* $F_1, F_2, \ldots, F_l$ *are distinct inequivalent representations over* K *of degrees* $f_1, f_2, \ldots, f_l$ *occurring with multiplicities* $e_1, e_2, \ldots, e_l$ *in* A. *Let* $\mathscr{C}(A)$ *be the commutant algebra of* A. *Then*
(i) $\dim \mathscr{C}(A) = e_1^2 + e_2^2 + \cdots + e_l^2$,
(ii) $\dim$ *centre* $\mathscr{C}(A) = l$,
(iii) $m = e_1 f_1 + e_2 f_2 + \cdots + e_l f_l$ *where* m *is the degree of* A.

Proof. (i)
$$A(x) = \sum_{i=1}^{l} \oplus (I_{e_i} \otimes F_i(x)) \tag{1.40}$$
is the short notation for
$A(x) = diag(\underbrace{F_1(x), \ldots, F_1(x)}_{e_1 times}, \underbrace{F_2(x), \ldots, F_2(x)}_{e_2 times}, \underbrace{F_l(x), \ldots, F_l(x)}_{e_l times})$,
where $F_1, F_2, \ldots, F_l$ are inequivalent irreducible representations of degrees $f_1, f_2, \ldots, f_l$, which occur with multiplicities $e_1, e_2, \ldots, e_l$ in A.
First we consider the special case,
$D(x) = diag(F(x), \ldots, F(x)) = I_e \otimes F(x)$, in which a single irreducible representation of degree f is repeated e times. The most general matrix T satisfying $TD(x) = D(x)T$ $(x \in G)$ is of the form
$$T = \begin{pmatrix} x_{11}I_f & x_{12}I_f & \ldots x_{1m}I_f \\ x_{21}I_f & x_{22}I_f & \ldots x_{2m}I_f \\ \ldots & \ldots & \ldots\ldots \\ x_{m1}I_f & x_{m2}I_f & \ldots x_{mm}I_f \end{pmatrix} = X \otimes I_f,$$
where $X = (x_{ij})$ is an arbitrary $e \times e$ matrix over K. Hence

we can say that $\mathscr{C}(D) = \mathscr{M}_e \otimes I_f$, where $\mathscr{M}_e$ is the complete matrix algebra over K of degree e. To verify that as an algebra, $\mathscr{C}(D)$ has the structure as $\mathscr{M}_e$.

Let $T_1 = X_1 \otimes I_f$ and $T_2 = X_2 \otimes I_f$ be two elements of $\mathscr{C}(D)$. Then, by the properties of the tensor product, $k_1T_1 + k_2T_2 = k_1(X_1 \otimes I_f) + k_2(X_2 \otimes I_f) = (k_1X_1 + k_2X_2) \otimes I_f$, and $T_1T_2 = (X_1 \otimes I_f)(X_2 \otimes I_f) = (X_1X_2) \otimes (I_fI_f) = (X_1X_2) \otimes I_f$. Thus we can state that

$$\mathscr{C}(D) \cong \mathscr{M}_e \tag{1.41}$$

Now consider the general case, $A(x) = \sum_{i=1}^{l} \oplus (I_{e_i} \otimes F_i(x))$.

If T is partitioned accordingly, all blocks of T which correspond to distinct summands of A are zero, while those which are associated with the same summand are given by $T = X \otimes I_f$. Hence $\mathscr{C}(A) = \sum_{i=1}^{l} (\mathscr{M}_{e_i} \otimes I_{f_i})$. The algebraic structure is described by $\mathscr{C}(A) = \sum_{i=1}^{l} \mathscr{M}_{e_i}$. $\mathscr{M}_{e_i}$ is a vector space over K of dimension e_i^2 . Since in a direct sum of vector spaces the dimensions are added, we get $dim\ \mathscr{C}(A) = e_1^2 + e_2^2 + \cdots + e_l^2$.

(ii) To show that $dim(centre\ \mathscr{C}(A)) = \ell$.

First consider the centre of $\mathscr{M}_e$, that is the set of matrices Z satisfying $ZX = XZ$ for all $e \times e$ matrices X. If $Z = (z_{ij})$ lies in the centre of $\mathscr{M}_e$, then $ZX = XZ$ holds when we take for X the matrix $W = diag(w_1, w_2, \ldots, w_e)$, where $w_1, w_2, \ldots, w_e$ are distinct elements of K.

$ZW = WZ$ implies that

$$\begin{pmatrix} z_{11}w_1 & z_{12}w_2 & \dots z_{1e}w_e \\ \dots & \dots & \dots\dots \\ z_{e1}w_1 & z_{e2}w_2 & \dots z_{ee}w_e \end{pmatrix} = \begin{pmatrix} w_1z_{11} & w_2z_{12} & \dots w_ez_{1e} \\ \dots & \dots & \dots\dots \\ w_1z_{e1} & w_2z_{e2} & \dots w_ez_{ee} \end{pmatrix}$$

$\implies z_{ij}w_j = w_iz_{ij}$ for $i, j = 1, 2, \dots, e$.

$\implies z_{ij} = 0$ for $i \neq j$.

$\implies z_{ij}w_j = w_iz_{ij}$ for $i, j = 1, 2, \dots, e$.

Thus Z must be a diagonal matrix.

Let $Z = diag(z_1, z_2, \dots, z_e)$. Next we choose for X the permutation matrix $P = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & 0 & \dots 0 \end{pmatrix}$.

Thus $ZP = PZ$ gives

$$\begin{pmatrix} z_1 & 0 & \dots & 0 \\ 0 & z_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & z_e \end{pmatrix} \begin{pmatrix} 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} z_1 & 0 & \dots & 0 \\ 0 & z_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & z_e \end{pmatrix}$$

$$\implies \begin{pmatrix} 0 & z_1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ z_e & 0 & \dots & 0 \end{pmatrix} = \begin{pmatrix} 0 & z_2 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ z_1 & 0 & \dots & 0 \end{pmatrix}$$

$\implies z_1 = z_2 = \dots = z_e = c$.

Thus $Z = cI_f$.

Next, we determine the centre of $\mathscr{C}(A)$. A typical element of $\mathscr{C}(A)$ is a matrix of the form $T = \sum_{i=1}^{l} \oplus(X_i \otimes I_{f_i})$ where X_i is an arbitrary $e_i \times e_i$ matrix.
Hence if $Z = \sum_{i=1}^{l} \oplus(C_i \otimes I_{f_i})$ lies in the centre of $\mathscr{C}(A)$, then $ZT = TZ$ for all T in $\mathscr{C}(A)$.
$ZT = \sum_{i=1}^{l} \oplus((C_i \otimes I_{f_i})(X_i \otimes I_{f_i})) = \sum_{i=1}^{l} \oplus(C_i X_i \otimes I_{f_i})$.
$TZ = \sum_{i=1}^{l} \oplus((X_i \otimes I_{f_i})(C_i \otimes I_{f_i})) = \sum_{i=1}^{l} \oplus(X_i C_i \otimes I_{f_i})$.
Hence $C_i X_i = X_i C_i$ and $C_i = c_i I_{e_i}$ for $i = 1, 2, \ldots, l$.
Thus we get $Z = \sum_{i=1}^{l} \oplus(c_i I_{e_i} \otimes I_{f_i}) = \sum_{i=1}^{l} \oplus c_i I_{e_i f_i}$.
In every algebra, the centre forms a subalgebra and hence a vector space. From the above expression, it is clear that the centre of $\mathscr{C}(A)$ is a vector space of dimension l.
(iii) To prove the result, we compare the degrees of both sides of equation 1.40.
We have $deg(A(x)) = m$.
The number of rows (columns) on the RHS is $e_1 f_1 + \cdots + e_l f_l$.
Hence we get $m = e_1 f_1 + \cdots + e_l f_l$. □

In module language, we get the following.
Let V be a completely reducible G-module over V which affords the representation $A(x)$. Then a matrix T satisfying $TA(x) = A(x)T$ corresponds to a G-endomorphism of V, that is a G-homomorphism $\theta : V \to V$. If θ_1 and θ_2 are G-endomorphisms, so also are $k_1\theta_1 + k_2\theta_2$ $(k_1, k_2 \in K)$ and $\theta_1\theta_2$, composition being defined in the usual way. Thus the set of all G-endomorphisms of V forms an algebra over K,

which is often denoted by $\mathscr{H}(V)$ or $Hom_K(V,V)$. We call $\mathscr{H}(V)$ the endomorphism algebra of V over K. Thus we have $\mathscr{C}(A) \cong \mathscr{H}(V)$. So the above theorem can be recast as follows.

Theorem 21. *Let V be a completely reducible G-module over K of dimension m, and suppose that $V = U_1 \oplus \cdots \oplus U_1 \oplus U_2 \oplus \cdots \oplus U_2 \oplus \cdots \oplus U_l \oplus \cdots \oplus U_l$, where $U_1, U_2, \ldots, U_l$ are mutually non-isomorphic irreducible G- modules. Let $dim U_i = f_i$ and let e_i be the multiplicity of U_i in V $(i = 1, 2, \ldots, l)$. Then if $\mathscr{H}(V)$ is the endomorphism algebra of V, we have that*
(i) dim $\mathscr{H}(V) = e_1^2 + e_2^2 + \cdots + e_l^2$,
(ii) dim centre $\mathscr{H}(V) = l$,
(iii) $m = e_1 f_1 + e_2 f_2 + \cdots + e_l f_l$.

1.9 Exercises

1. Let H be a subgroup of G with coset representatives $t_1, t_2, \ldots, t_n$. Show that the kernel of the permutation representation $\sigma(x) = \begin{pmatrix} Ht_1 & Ht_2 & \ldots & Ht_n \\ Ht_1x & Ht_2x & \ldots & Ht_nx \end{pmatrix}$ is the group $\bigcap_{i=1}^{n} t_i^{-1} H t_i$.

2. Show that the map, $a \to C(a) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ defines a representation of the cyclic group gp $\{a | a^3 = 1\}$. Prove that this representation is irreducible over the

field of real numbers.

3. Let ϵ be a linear map of the m-dimensional vector space V into itself such that $\epsilon^2 = \epsilon$, $\epsilon \neq 0$, $\epsilon \neq i$. Define $U = \{\mathbf{u} : \mathbf{u}\epsilon = \mathbf{u}\}$ and $W = \{\mathbf{w} : \mathbf{w}\epsilon = \mathbf{0}\}$. Show that U and W are subspaces of V which are invariant under ϵ. Prove that $V = U \oplus W$. Deduce that if E is an $m \times m$ matrix such that $E^2 = E$, $E \neq 0$, $E \neq I$, there exists an integer r satisfying $1 \leq r < m$ and a non-singular matrix T such that $T^{-1}ET = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = J$.

4. Let G be a group. Suppose that with each $x \in G$ there is associated a matrix $A(x)$ in such a way that $A(x)A(y) = A(xy)$; it is not assumed that $A(x)$ is non-singular. Prove that if $A(1)$ is singular but non-zero, then $A(x) \sim \begin{pmatrix} B(x) & 0 \\ 0 & 0 \end{pmatrix}$ where $B(x)$ is a representation of G (consisting of non-singular matrices).

5. Let G be a permutation group of degree m (≥ 2). Suppose that $V = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m]$ affords the natural representation of G, that is $\mathbf{v}_i\sigma = \mathbf{v}_{i\sigma}$, ($\sigma \in G; i = 1, 2, \ldots, m$). Put $\mathbf{u}_j = \mathbf{v}_j - \mathbf{v}_m$ ($j = 1, 2, \ldots, m-1$). Show that $U = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{m-1}]$ is an $(m-1)$-dimensional G-module. In particular, when $m = 4$, compute the 3×3 matrices that describe the action of $\tau = (12), \rho = (123), \lambda = (12)(34), \gamma = (1234)$ on U and obtain the character value in each case.

6. Suppose that the group G has a matrix representation $A(x)$ of degree two over the rationals, with the property that, for a certain central element z of G $A(z) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Prove that A is reducible over the rationals.

Solutions to Exercises

1. The element u in G belongs to the kernel of σ if and only if $Ht_iu = Ht_i$ $(i = 1, 2, \ldots, n)$
   if and only if $t_iu \in Ht_i$ $(i = 1, 2, \ldots, n)$
   if and only if $u \in t_i^{-1}Ht_i$ $(i = 1, 2, \ldots, n)$
   if and only if $u \in \bigcap_{i=1}^{n} t_i^{-1}Ht_i$.
   Therefore Kernel of σ is the group $\bigcap_{i=1}^{n} t_i^{-1}Ht_i$.

2. It is sufficient to show that $[C(a)]^3 = I$.
   $[C(a)]^2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$.
   $[C(a)]^3 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$.
   If the representation was reducible over R, there would exist a real matrix T such that $T^{-1}C(a)T = \begin{pmatrix} \alpha & 0 \\ \gamma & \beta \end{pmatrix}$ where $\alpha, \beta, \gamma \in R$. Then α and β would be the latent roots of $C(a)$. The latent roots of $C(a)$ are given by

$$|\lambda I - C(a)| = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda+1 \end{vmatrix} = \lambda(\lambda+1)+1 = \lambda^2+1.$$

$\lambda^2+1=0 \implies \lambda = \dfrac{-1 \pm i\sqrt{3}}{2}$.

Since the latent roots are complex, we obtain that this representation is irreducible over R.

3. If $\mathbf{u}_1$, $\mathbf{u}_2 \in U$ and a_1, a_2 are scalars, then $(a_1\mathbf{u}_1 + a_2\mathbf{u}_2)\epsilon = a_1\mathbf{u}_1\epsilon + a_2\mathbf{u}_2\epsilon = a_1\mathbf{u}_1 + a_2\mathbf{u}_2$. Thus $a_1\mathbf{u}_1 + a_2\mathbf{u}_2 \in U$. Hence U is a subspace.

   As W is the kernel of ϵ, W is a subspace of V.

   Also if $\mathbf{u} \in U$, then $\mathbf{u}\epsilon \in U$ because $(\mathbf{u}\epsilon)\epsilon = \mathbf{u}\epsilon^2 = \mathbf{u}\epsilon = \mathbf{u}$. Therefore U is invariant under ϵ. Similarly for $\mathbf{w} \in W$, $\mathbf{w}\epsilon = \mathbf{0} \in W$ and therefore W is invariant under ϵ.

   For an arbitrary $\mathbf{v} \in V$, we can write $\mathbf{v} = \mathbf{v}\epsilon + \mathbf{v}(i-\epsilon)$. Now $(\mathbf{v}\epsilon)\epsilon = \mathbf{v}\epsilon^2 = \mathbf{v}\epsilon$. Thus $\mathbf{v}\epsilon \in U$ and $\mathbf{v}(i-\epsilon)\epsilon = \mathbf{v}\epsilon - \mathbf{v}\epsilon^2 = \mathbf{v}\epsilon - \mathbf{v}\epsilon = \mathbf{0}$. Thus $\mathbf{v}(i-\epsilon) \in W$. Hence $V = U+W$. But $U \cap W = \{\mathbf{0}\}$, because $\mathbf{u} = \mathbf{w}$ implies that $\mathbf{u}\epsilon = \mathbf{w}\epsilon$, that is $\mathbf{u} = \mathbf{0}$.

   Given an $m \times m$ matrix E, let $V = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m]$ be the space of m-dimensional row vectors of E. Define ϵ by $\mathbf{v}_i\epsilon = \mathbf{v}_i E$ $(i = 1, 2, \ldots, m)$. Then we obtain two subspaces U and W defined by $U = \{\mathbf{u} | \mathbf{u}\epsilon = \mathbf{u}\}$ and $W = \{\mathbf{w} | \mathbf{w}\epsilon = \mathbf{0}\}$, which are invariant under ϵ. Also $V = U \oplus W$.

   Choose a basis of V adapted to U and W, say $V' = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{m-r}]$. Relative to this ba-

sis, $\mathbf{u}_i\epsilon = \mathbf{u}_i$ and $\mathbf{w}_j\epsilon = \mathbf{0}$ and so the matrix of ϵ is given by the matrix $J = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$. If T is the matrix which describes the change of basis, then T is non-singular and $J = T^{-1}ET$.

4. Let $E = A(1)$. Then since $[A(1)]^2 = A(1)$, $E^2 = E$ and $E \neq 0$, $E \neq I$ ($A(1) \neq 0$, but is singular). By the above Exercise, there exist $r \in Z$, $1 \leq r < m$ such that $E \sim J = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$. Let $E = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = A(1)$. Also let $A(x) = \begin{pmatrix} B(x) & B_1(x) \\ B_2(x) & B_3(x) \end{pmatrix}$ where $B(x)$ is an $r \times r$ matrix.
$A(x) = A(1)A(x)A(1)$ $(x \in G)$ which implies
$$\begin{pmatrix} B(x) & B_1(x) \\ B_2(x) & B_3(x) \end{pmatrix} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} B(x) & B_1(x) \\ B_2(x) & B_3(x) \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} B(x) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} B(x) & 0 \\ 0 & 0 \end{pmatrix}.$$
Thus $B_1(x) = B_2(x) = B_3(x) = 0$.

5. If $\sigma \in G$, then $\mathbf{u}_j\sigma = (\mathbf{v}_j - \mathbf{v}_m)\sigma = \mathbf{v}_j\sigma - \mathbf{v}_m\sigma = (\mathbf{v}_j\sigma - \mathbf{v}_m) - (\mathbf{v}_m\sigma - \mathbf{v}_m) \in U$, that is U is a G-module.
To show that $U = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{m-1}]$
The vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{m-1}$ are linearly independent.
To prove this, consider $c_1\mathbf{u}_1 + \cdots + c_{m-1}\mathbf{u}_{m-1} = \mathbf{0}$.
i.e. $c_1(\mathbf{v}_1 - \mathbf{v}_m) + \cdots + c_{m-1}(\mathbf{v}_{m-1} - \mathbf{v}_m) = \mathbf{0}$.

i.e. $c_1\mathbf{v}_1 + \cdots + c_{m-1}\mathbf{v}_{m-1} - (c_1 + \cdots + c_{m-1})\mathbf{v}_m = \mathbf{0} \implies c_1 = \cdots = c_{m-1} = 0$ (as $v_1, \ldots v_{m-1}$ are linearly independent)

Thus $dim\ U = m - 1$. If A is the matrix representation afforded by τ with character ϕ, we find that $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)\tau = (\mathbf{v}_1 - \mathbf{v}_4, \mathbf{v}_2 - \mathbf{v}_4, \mathbf{v}_3 - \mathbf{v}_4)\tau = (\mathbf{v}_2 - \mathbf{v}_4, \mathbf{v}_1 - \mathbf{v}_4, \mathbf{v}_3 - \mathbf{v}_4) = (\mathbf{u}_2, \mathbf{u}_1, \mathbf{u}_3)$.

$$A(\tau) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \phi(\tau) = 1.$$

$(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)\rho = (\mathbf{v}_1 - \mathbf{v}_4, \mathbf{v}_2 - \mathbf{v}_4, \mathbf{v}_3 - \mathbf{v}_4)\rho = (\mathbf{v}_2 - \mathbf{v}_4, \mathbf{v}_3 - \mathbf{v}_4, \mathbf{v}_1 - \mathbf{v}_4) = (\mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_1)$.

$$A(\rho) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } \phi(\rho) = 0.$$

$(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)\lambda = (\mathbf{v}_1 - \mathbf{v}_4, \mathbf{v}_2 - \mathbf{v}_4, \mathbf{v}_3 - \mathbf{v}_4)\lambda = (\mathbf{v}_2 - \mathbf{v}_3, \mathbf{v}_1 - \mathbf{v}_3, \mathbf{v}_4 - \mathbf{v}_3) = (\mathbf{v}_2 - \mathbf{v}_4 - (\mathbf{v}_3 - \mathbf{v}_4), \mathbf{v}_1 - \mathbf{v}_4 - (\mathbf{v}_3 - \mathbf{v}_4), -(\mathbf{v}_3 - \mathbf{v}_4)) = (\mathbf{u}_2 - \mathbf{u}_3, \mathbf{u}_1 - \mathbf{u}_3, -\mathbf{u}_3)$.

$$A(\lambda) = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } \phi(\lambda) = -1.$$

$(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)\gamma = (\mathbf{v}_1 - \mathbf{v}_4, \mathbf{v}_2 - \mathbf{v}_4, \mathbf{v}_3 - \mathbf{v}_4)\gamma = (\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1, \mathbf{v}_4 - \mathbf{v}_1) = (\mathbf{v}_2 - \mathbf{v}_4 - (\mathbf{v}_1 - \mathbf{v}_4), \mathbf{v}_3 - \mathbf{v}_4 - (\mathbf{v}_1 - \mathbf{v}_4), -(\mathbf{v}_1 - \mathbf{v}_4)) = (\mathbf{u}_2 - \mathbf{u}_1, \mathbf{u}_3 - \mathbf{u}_1, -\mathbf{u}_1)$.

$$A(\gamma) = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \text{ and } \phi(\gamma) = -1.$$

6. By Schur's lemma, if A were irreducible over the rationals, then any rational matrix T satisfying $TA(x) = A(x)T$ for all $x \in G$ would have to be either zero or else non-singular.
   Given z is a central element of G.
   $\implies zx = xz$ for all $x \in G$.
   $\implies A(zx) = A(xz)$ for all $x \in G$.
   $\implies A(z)A(x) = A(x)A(z)$ for all $x \in G$.
   Consider the matrix $T = I + A(z) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.
   Then T is singular and $TA(x) = (I + A(z))A(x) = A(x) + A(z)A(x) = A(x) + A(x)A(z) = A(x)(I + A(z)) = A(x)T$ for all $x \in G$. This proves that A must be reducible.

Chapter 2

Elementary Properties of Group Characters

2.1 Orthogonality relations

From now on we assume that G is a finite group of order g and that K is the field of complex numbers. Hence by Maschke's Theorem, all representations of G are completely reducible.

Let $A(x)$ and $B(x)$ be inequivalent irreducible representations of degrees f and f' respectively. Then,

$$A(x) = (a_{ij}(x)) \quad \text{for} \quad 1 \leq i, j \leq f,$$
$$B(x) = (b_{pq}(x)) \quad \text{for} \quad 1 \leq p, q \leq f'.$$

Now choose $f'f$ arbitrary complex numbers ξ_{pj} and arrange them in an $f' \times f$ matrix, $\Xi = (\xi_{pj})$. (Ξ to be read as Xi; i.e. it is capital xi)
Next we construct the matrix $C == \sum_{y \in G} B(y^{-1}) \Xi A(y)$.

Let x be a fixed element of G. Then $z = yx$ ranges over G when y does. Thus we write

$$\begin{aligned} C &= \sum_{z \in G} B(z^{-1}) \Xi A(z) \\ &= \sum_{y \in G} B(x^{-1}y^{-1}) \Xi A(yx) \\ &= B(x^{-1}) \left(\sum_{y \in G} B(y^{-1}) \Xi A(y) \right) A(x) \\ \implies C &= B(x^{-1}) C A(x) \\ &= [B(x)]^{-1} C A(x) \\ \implies B(x)C &= CA(x) \quad \forall x \in G. \end{aligned}$$

Hence by applying Schur's lemma, we get C is the zero matrix.

$$\begin{aligned} & i.e. \sum_{y \in G} B(y^{-1}) \Xi A(y) = 0 \\ \implies & \sum_{y \in G} \sum_{q,i} b_{pq}(y^{-1}) \xi_{qi} a_{ij}(y) = 0, \quad \forall p, j. \end{aligned}$$

Since these equations hold for an arbitrary choice of Ξ, the coefficient of each ξ_{qi} must be zero.

$$i.e. \sum_{y \in G} b_{pq}(y^{-1}) a_{ij}(y) = 0 \quad \forall p, q, i, j. \tag{2.1}$$

We introduce the notion of an inner product for functions on G with values in K. Let $\phi(x)$ and $\psi(x)$ be two such functions and write

$$\langle \phi, \psi \rangle = \frac{1}{g} \sum_{x \in G} \phi(x)\psi(x^{-1}).$$

Since this sum is unaltered if we replace x by x^{-1}, it follows that

$$\langle \phi, \psi \rangle = \langle \psi, \phi \rangle.$$

We say that the functions $\phi(x)$ and $\psi(x)$ are orthogonal if $\langle \phi, \psi \rangle = 0$. Therefore equation (2.1) can be expressed as

$$\langle a_{ij}, b_{pq} \rangle = 0 \quad \forall p, q, i, j.$$

Thus if $A(x)$ and $B(x)$ are inequivalent irreducible representations, then every element of the matrix $A(x)$, when regarded as a function on G, is orthogonal to every element of $B(x)$.

Next, we consider a single irreducible representation of degree f. Then, as before, the matrix

$$C = \sum_{y \in G} A(y^{-1}) \Xi A(y) \tag{2.2}$$

satisfies $A(x)C = CA(x)\ \forall x \in G$. So by the corollary to Schur's Lemma,

$$C = \lambda I_f \tag{2.3}$$

where $\lambda = \lambda\Big(\Xi\Big)$ is a scalar depending on the matrix Ξ. To evaluate λ, take the trace on both sides of equation (2.3). Thus,

$$tr(C) = f\lambda \tag{2.4}$$

Since $A(y^{-1}) = (A(y))^{-1}$, it follows that

$$tr\Bigg(A(y^{-1})\Xi A(y)\Bigg) = tr(\Xi).$$

From equation (2.2), it follows that

$$tr(C) = g\ tr(\Xi) \tag{2.5}$$

From equation (2.4) and (2.5), we get $\lambda = \dfrac{g}{f} tr(\Xi)$. Therefore, equating (2.2) and (2.3) for C

$$\sum_{y\in G} A(y^{-1})\Xi A(y) = \frac{g}{f}(\xi_{11} + \xi_{22} + \cdots + \xi_{ff})I_f.$$

Comparing the coefficients of ξ_{jp}, we obtain that

$\sum_{y\in G} a_{ij}(y^{-1})a_{pq}(y) = 0$ if either $i \neq q$ or $j \neq p$ and

$\sum_{y\in G} a_{ij}(y^{-1})a_{ji}(y) = \dfrac{g}{f} \qquad \forall i, j.$

$$i.e. \langle a_{ij}, a_{pq}\rangle = \frac{1}{f}\delta_{iq}\delta_{jp}. \tag{2.6}$$

Theorem 22. *(Character relations of the first kind): Let $\chi(x)$ and $\chi'(x)$ be the characters of the irreducible representations $A(x)$ and $B(x)$ respectively. Then, $\langle \chi, \chi' \rangle$ =*

$$\begin{cases} 1, & \text{if } A(x) \sim B(x) \\ 0, & \text{if } A(x) \nsim B(x) \end{cases}$$

Proof. Let

$$A(x) = (a_{ij}(x)) \quad \text{for} \quad 1 \le i, j \le f$$
$$B(x) = (b_{pq}(x)) \quad \text{for} \quad 1 \le p, q \le f'.$$

So

$$\chi(x) = tr(A(x)) = a_{11}(x) + a_{22}(x) + \cdots + a_{ff}(x),$$
$$\chi'(x) = tr(B(x)) = b_{11}(x) + b_{22}(x) + \cdots + b_{f'f'}(x).$$

If $A(x)$ and $B(x)$ are inequivalent,

$$\begin{aligned} \langle \chi, \chi' \rangle &= \frac{1}{g} \sum_{x \in G} \chi(x)\chi'(x^{-1}) \\ &= \frac{1}{g} \sum_{x \in G} (a_{11}(x) + \cdots + a_{ff}(x))(b_{11}(x^{-1}) + \cdots + b_{f'f'}(x^{-1})) \\ &= \sum_{i,p} \langle a_{ii}, b_{pp} \rangle = 0. \end{aligned}$$

If A and B are equivalent, we may assume that $A = B$ because equivalent representations have same character. Then by equation (2.6),

$$\begin{aligned} \langle \chi, \chi' \rangle &= \sum_{i,p=1}^{f} \langle a_{ii}, a_{pp} \rangle \\ &\frac{1}{f} \sum_{i,p=1}^{f} \delta_{ip}\delta_{ip} \\ &= \frac{1}{f} f = 1. \end{aligned}$$

□

Definition 23. *The character of an irreducible representation is called a* ***simple character*** *while the character of a reducible representation is termed* ***compound****.*

If G is a group of order g, then any function ϕ on G may be regarded as a g-tuple. Thus $\phi = (\phi(x_1), \phi_(x_2), \ldots, \phi(x_g))$ where $x_1, x_2, \ldots, x_g$ are the elements of g enumerated in some fixed manner.
When viewed as g-tuples, any collection $\chi^{(1)}, \chi^{(2)}, \ldots, \chi^{(s)}$ of distinct simple characters is linearly independent. So, let $\sum_{i=1}^{s} c_i\chi^{(i)} = 0$, where 0 in the RHS is the g-tuple zero. Taking the inner product of this equation with $\chi^{(j)}$ for $1 \leq j \leq s$,

$$\left\langle \sum_{i=1}^{s} c_i\chi^{(i)}, \chi^{(j)} \right\rangle = 0$$

$$\implies \sum_{i=1}^{s} c_i \left\langle \chi^{(i)}, \chi^{(j)} \right\rangle = 0$$

$$\implies c_j = 0 \quad \forall 1 \leq j \leq s.$$

Therefore, $\chi^{(1)}, \chi^{(2)}, \cdots, \chi^{(s)}$ are linearly independent. Since there can be no set comprising of more than g linearly independent g-tuples, we get $s \leq g$. Hence it follows that a group of order g can have at most g inequivalent irreducible representations. We shall denote the precise number of irreducible representations by r. Let $\chi^{(1)}, \chi^{(2)}, \ldots, \chi^{(r)}$ be the complete set of simple characters and suppose that they correspond to the irreducible representations

$$F^{(1)}, F^{(2)}, \ldots, F^{(r)} \tag{2.7}$$

which are determined only up to equivalence. The degree of the irreducible representations will be denoted by $f^{(1)}, f^{(2)}, \ldots, f^{(r)}$ respectively. By Maschke's Theorem, if A is an arbitrary representation, then $A \sim diag(F, F', F'', \ldots)$ where $F, F', F'', \ldots$ are irreducible. The constituents of A need not be distinct but each of them can be chosen to be one of the complete set given in (2.7).

If ϕ is the character of A, we can write

$$\phi = \sum_{j=1}^{r} d_j \chi^{(j)} \tag{2.8}$$

where $d_j \geq 0$ is the multiplicity of $F^{(j)}$ in A. Taking the inner product with $\chi^{(i)}$,

$$\begin{aligned} \langle \phi, \chi^{(i)} \rangle &= \left\langle \sum_{j=1}^{r} d_j \chi^{(j)}, \chi^{(i)} \right\rangle \\ &= \sum_{j=1}^{r} d_j \langle \chi^{(j)}, \chi^{(i)} \rangle \\ &= d_i. \end{aligned}$$

Thus we have obtained

$$d_i = \langle \phi, \chi^{(i)} \rangle \quad \text{for} \quad 1 \leq i \leq r. \tag{2.9}$$

Since this is analogous to the way in which the coefficients of a Fourier series are determined, equation (2.8) is called the Fourier analysis of ϕ or A.

Theorem 24. *Two representations of a finite group over the*

complex field are equivalent if and only if they have the same character.

Proof. Necessary part is already proved.
Sufficient part: Let A and B be two representations of the finite group G over the complex field. Let ϕ be the character of A, ψ be the character of B. If $\chi^{(1)}, \chi^{(2)}, \ldots, \chi^{(r)}$ are the complete set of simple characters of the group G, then we can write $\phi = \sum_{j=1}^{r} d_j \chi^{(j)}$ and $\psi = \sum_{j=1}^{r} e_j \chi^{(j)}$, where d_j and e_j are the multiplicities of the irreducible representations corresponding to the simple characters in A and B respectively. Now, suppose that $\phi(x) = \psi(x)\ \forall x \in G$. Since we have $d_i = \langle \phi, \chi^{(i)} \rangle$ and $e_i = \langle \psi, \chi^{(i)} \rangle$, for $i = 1, 2, \ldots, r$, it follows that $d_i = e_i$ for $i = 1, 2, \ldots, r$. This means that A and B are equivalent to the same diagonal array of irreducible constitutes and are therefore equivalent to each other. □

Note. *Thus the trace of a representation furnishes us with complete information about the irreducible constituents into which this representation may be decomposed. So the character characterizes a representation up to equivalence. Let* $\phi = \sum_{j=1}^{r} d_j \chi^{(j)}$ *and* $\psi = \sum_{j=1}^{r} e_j \chi^{(j)}$. *Then*

$$\begin{aligned}
\langle \phi, \psi \rangle &= \frac{1}{g} \sum_{x \in G} \phi(x)\psi(x^{-1}) \\
\implies \langle \phi, \psi \rangle &= \frac{1}{g} \sum_{x \in G} \sum_{i=1}^{r} d_i \chi^{(i)}(x) \sum_{j=1}^{r} e_j \chi^{(j)}(x^{-1}) \\
&= \frac{1}{g} \sum_{x \in G} \sum_{i,j=1}^{r} d_i e_j \chi^{(i)}(x)\chi^{(j)}(x^{-1}) \\
&= \sum_{i,j=1}^{r} d_i e_j \langle \chi^{(i)}\chi^{(j)} \rangle \\
&= \sum_{j=1}^{r} d_j e_j.
\end{aligned}$$

In particular, $\langle \phi, \phi \rangle = \sum_{j=1}^{r} d_j^2$. *Thus we get the following proposition.*

Proposition 25. *A representation with character* ϕ *is irreducible if and only if* $\langle \phi, \phi \rangle = 1$.

Proof. We have $\langle \phi, \phi \rangle = \sum_{j=1}^{r} e_j^2$ if $\phi = \sum_{j=1}^{r} e_j \chi^{(j)}$. A representation is irreducible if and only if it consists of a single un repeated irreducible constituent. Thus, we get ϕ is irreducible if and only if $\langle \phi, \phi \rangle = 1$. □

Example 26. *Consider the representation* D *of* S_3.

$D(1) = I$, $D(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $D(b) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$,

$D(c) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$, $D(e) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$,

$D(f) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$.

Let ϕ denote the character of the representation D. Then,
$\phi(I) = 2$,
$\phi\Big(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\Big) = \phi\Big(\begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}\Big) = \phi\Big(\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}\Big) = 0$,
$\phi\Big(\begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix}\Big) = -1$, $\phi\Big(\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}\Big) = -1$, *and*
$\phi(D(1)^{-1}) = 2$, $\phi(D(a)^{-1}) = 0$, $\phi(D(b)^{-1}) = 0$, $\phi(D(c)^{-1}) = 0$,
$\phi(D(e)^{-1}) = -1$, $\phi(D(f)^{-1}) = -1$.

Therefore,

$$\langle \phi, \phi \rangle = \frac{1}{g} \sum_{x \in S_3} \phi(x)\phi(x^{-1}) \tag{2.10}$$

$$= \frac{1}{3!}[2 \times 2 + 0 + 0 + 0 + (-1) \times (-1) + (-1) \times (-1)] = 1 \tag{2.11}$$

Thus the representation D is irreducible over the complex field.

2.2 Group Algebra

Let us introduce a new concept which connects the structure of group G with that of the underlying field $K = \mathbb{C}$. Let $G : x_1(= 1), x_2, \ldots, x_g$ be a finite multiplicative group. Regard the elements of G as the basis vectors of a vector space

$$G_{\mathbb{C}} = [x_1, x_2, \ldots, x_g] \tag{2.12}$$

over the complex numbers. Thus a typical element of $G_{\mathbb{C}}$ is a formal sum $v = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_g x_g$ where $\alpha_1, \alpha_2, \ldots, \alpha_g \in \mathbb{C}$ are arbitrary. Let $w = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_g x_g$ where $\beta_1, \beta_2, \ldots, \beta_g \in \mathbb{C}$ be another element of $G_{\mathbb{C}}$.

Then $v = w$ if and only if $\alpha_i = \beta_i$, $i = 1, 2, 3, \ldots, g$. We have the usual rules for a vector space namely $v+w = \sum_{i=1}^{g}(\alpha_i+\beta_i)x_i$ and $\lambda v = \sum_{i=1}^{g} \lambda\alpha_i x_i$, where $\lambda \in \mathbb{C}$. Also $1 \cdot v = v$. Further, $G_{\mathbb{C}}$ possesses a multiplicative structure given as follows. Suppose that the multiplication table for G is expressed as

$$x_i x_j = x_{\mu(i,j)} \tag{2.13}$$

where $\mu(i, j)$ is a well-defined integer lying between 1 and g. Then multiplication in $G_{\mathbb{C}}$ is defined by $vw = \sum_{i,j=1}^{g} \alpha_i\beta_j x_{\mu(i,j)}$. This multiplication is associative since multiplication in G is associative. Since $G_{\mathbb{C}}$ is now a vector pace which is endowed with an associative multiplication, we call $G_{\mathbb{C}}$ the group algebra of G over $\mathbb{C}$. Clearly $G_{\mathbb{C}}x \subset G_{\mathbb{C}} \quad (\forall x \in G)$ and hence $G_{\mathbb{C}}$ is a G-module of degree g over $\mathbb{C}$. Let $R(x)$ be the representation afforded by the $G-$module $G_{\mathbb{C}}$. If we use the natural basis (2.12), the action of x is described by $[x_1, x_2, \ldots, x_g]x = [x_1x, x_2x, \ldots, x_gx]$ which is identical with the permutation representation $\pi(x)$ mentioned earlier.

Hence $G_{\mathbb{C}}$ as a G- module affords the right regular representation of G. If $R(x) = (r_{ij}(x)) \quad (i, j = 1, 2, \ldots, g)$, then $x_i x = \sum_{j=1}^{g} r_{ij}(x)x_j \quad (i = 1, 2, \ldots, g)$. Put $x = x_s$ and use equation (2.13). Then $x_i x_s = \sum_{j=1}^{g}(r_{ij}(x_s))x_j$ and from equation (2.13), $x_i x_s = x_{\mu(i,s)}$. Hence it follows that

$$r_{ij}(x_s) = \delta_{\mu(i,s),j}.$$

Example 27. *Find the right-regular representation of the four-group:* $V = 1, a, b, ab$ *where* $a^2 = b^2 = 1$, $ab = ba$.

Solution.

$$\begin{bmatrix} 1 & a & b & ab \end{bmatrix} 1 = \begin{bmatrix} 1 & a & b & ab \end{bmatrix}$$
$$\begin{bmatrix} 1 & a & b & ab \end{bmatrix} a = \begin{bmatrix} a & 1 & ab & b \end{bmatrix}$$
$$\begin{bmatrix} 1 & a & b & ab \end{bmatrix} b = \begin{bmatrix} b & ab & 1 & a \end{bmatrix}$$
$$\begin{bmatrix} 1 & a & b & ab \end{bmatrix} ab = \begin{bmatrix} ab & b & a & 1 \end{bmatrix}$$

$$R(1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; R(a) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$R(b) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \; R(ab) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

$$R(a)R(b) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = R(ab).$$

Denote the character of $R(x)$ by $\rho(x)$. Therefore, $\rho(x) = \sum_{i=1}^{g} r_{ii}(x)$. We have already seen that $\rho(x)$ is the number of symbols fixed by x; i.e. for a particular x, $\rho(x)$ is the number of times that $x_i x = x_i$ $(i = 1, 2, 3, \ldots, g)$ holds.
Here this is possible only when $x = 1$. Hence $r_{ii}(x) = 0$ if $x \neq 1$ and $r_{ii}(x) = 1$ when $x = 1$ for $i = 1, 2, 3, \ldots, g$.
Thus $\rho(1) = g$, $\rho(x) = 0$ for all $x \neq 1$. So the character of R is given by the g-tuple $\rho = (g, 0, 0, \ldots, 0)$. Now suppose that

$$\rho = \sum_{i=1}^{r} e_i \chi^{(i)} \tag{2.14}$$

is the Fourier analysis of ρ. Then,

$$\begin{aligned} e_i &= \langle \rho, \chi^{(i)} \rangle \\ &= \frac{1}{g} \sum_{j=1}^{g} \rho(x_j) \chi^{(i)}(x_j^{-1}) \\ &= \frac{1}{g} g \chi^{(i)}(1) \\ &= \chi^{(i)}(1) = f^{(i)} \end{aligned}$$

i.e. $e_i = f^{(i)}$ $(i = 1, 2, \ldots, r)$. So, (2.14) become

$$\rho(x) = \sum_{i=1}^{r} f^{(i)} \chi^{(i)} \tag{2.15}$$

which means that $R(x) \sim \sum_{i=1}^{r} \oplus \left(I_{f^{(i)}} \otimes F^{(i)}(x) \right)$. This is the decomposition of $R(x)$ into irreducible constituents. Put $x = 1$ in (2.15). $\rho(1) = \sum_{i=1}^{r} \left[f^{(i)} \right]^2$

$$\implies g = \sum_{i=1}^{r} \left(f^{(i)}\right)^2. \tag{2.16}$$

Consider the algebra $G_{\mathbb{C}}$ and the endomorphism algebra $\mathscr{H} = Hom(G_{\mathbb{C}}, G_{\mathbb{C}})$. We shall compute the dimension of $\mathscr{H}$ and the dimension of the centre of $\mathscr{H}$. $\mathscr{H}$ consists of all those linear maps $\theta : G_{\mathbb{C}} \to G_{\mathbb{C}}$ which commute with the action of G; i.e.

$$(v\theta)x = (vx)\theta; \quad v \in G_{\mathbb{C}}, \quad x \in G \tag{2.17}$$

Suppose that the image of 1 under θ is the element t of $G_{\mathbb{C}}$; i.e. $1\theta = t$.

Substituting $v = 1$ in (2.17), and leaving x arbitrary, $(1\theta)x = x\theta$ where $x \in G$, $t \in G_{\mathbb{C}}$. Hence by applying this formula to each $v \in G_{\mathbb{C}}$, we get

$$tv = v\theta, \quad v \in G_{\mathbb{C}}, t \in G_{\mathbb{C}}. \tag{2.18}$$

Conversely, if θ is defined by (2.18); i.e. if $v\theta = tv$, $\forall v \in G_{\mathbb{C}}$, where t is an arbitrary element of $G_{\mathbb{C}}$, we may replace v by vx in (2.18) and obtain $(vx)\theta = t(vx) = (tv)x = (v\theta)x$ so that (2.17) holds. Thus (2.18) sets up a one to one correspondence between the elements of $\mathscr{H}$ and those of $G_{\mathbb{C}}$. Every endomorphism of the G-module $G_{\mathbb{C}}$ is equivalent to the left multiplication by a fixed element of $G_{\mathbb{C}}$.

Theorem 28. *Let G be a group of order g. If G has k conjugacy classes, there are up to equivalence, k distinct irreducible representations over $\mathbb{C}$, $F^{(1)}, F^{(2)}, \ldots, F^{(k)}$ and if*

$f^{(i)}$ is the degree of $F^{(i)}$, then $g = \sum_{i=1}^{k} \left(f^{(i)}\right)^2$.

Proof. Let $G : x_1(=1), x_2, \ldots, x_g$, and $G_{\mathbb{C}} = [x_1, x_2, \ldots, x_g]$ be the vector space over $\mathbb{C}$ with elements of G as the basis vectors. Then $G_{\mathbb{C}}$ possesses a multiplicative structure defined as follows: If $x_i x_j = x_{\mu(i,j)}$, then for $v = \sum_{i=1}^{g} \alpha_i x_i$ and $w = \sum_{i=1}^{g} \beta_i x_i$, $vw = \sum_{i,j=1}^{g} \alpha_i \beta_j x_{\mu(i,j)}$. This multiplication is associative by virtue of the associativity for G. Hence $G_{\mathbb{C}}$ is an algebra referred to as the group algebra. Further since $G_{\mathbb{C}} x \subset G_{\mathbb{C}} \quad \forall x \in G$, $G_{\mathbb{C}}$ is a G- module of degree g over $\mathbb{C}$. Let $R(x)$ be the representation afforded by $G_{\mathbb{C}}$. Since $(x_1, x_2, \ldots, x_g)x = (x_1 x, x_2 x, \ldots, x_g x)$ is identical with the permutation representation $\pi(x)$, $G_{\mathbb{C}}$ as a G- module affords the right -regular representation of G.

Denoting the character of $R(x)$ by $\rho(x)$, we have $\rho(x)$ is the number of symbols fixed by x and hence ρ is given by the $g-$tuple $\rho = (g, 0, \ldots, 0)$. Also from the Fourier analysis of ρ,

$$\rho = \sum_{i=1}^{r} e_i \chi^{(i)}, \quad \text{where} \quad e_i = \langle \rho, \chi^{(i)} \rangle$$

$$= \frac{1}{g} \sum_{y \in G} \rho(y) \chi^{(i)}(y^{-1})$$

$$= \frac{1}{g} \rho(x_1) \chi^{(i)}(x_1)$$

$$= \frac{1}{g} g \chi^{(i)}(1)$$

$$= f^{(i)}$$

for $i = 1, 2, \ldots, r$.

$$\rho(x) = \sum_{i=1}^{r} f^{(i)} \chi^{(i)}(x) \tag{2.19}$$

$\implies R(x) = \sum_{i=1}^{r} \oplus \left[I_{f^{(i)}} \otimes F^{(i)}(x) \right]$. Also from equation (2.19), putting $x = 1$, $\rho(1) = \sum_{i=1}^{r} \left[f^{(i)} \right]^2$.

Consider the algebra $\mathscr{H} = Hom(G_{\mathbb{C}}, G_{\mathbb{C}})$. We will compute the dimension of $\mathscr{H}$ and the dimension of the centre of $\mathscr{H}$. $\mathscr{H}$ consists of all those linear maps $\theta : G_{\mathbb{C}} \to G_{\mathbb{C}}$ which commute with the action of G; i.e.

$$(v\theta)x = (vx)\theta, \quad \text{where} \quad v \in G_{\mathbb{C}}, x \in G. \tag{2.20}$$

Suppose that the image of 1 under θ is the element t of $G_{\mathbb{C}}$; i.e. $1\theta = t$.

Substituting $v = 1$ in (2.20), and leaving x arbitrary, $(1\theta)x = x\theta$ where $x \in G$, $t \in G_{\mathbb{C}}$. Hence by applying this formula to each $v \in G_{\mathbb{C}}$, we get

$$tv = v\theta, \quad v \in G_{\mathbb{C}}, t \in G_{\mathbb{C}}. \tag{2.21}$$

Conversely, if θ is defined by (2.21), i.e. if $v\theta = tv$, $\forall v \in G_{\mathbb{C}}$, were t is an arbitrary element of $G_{\mathbb{C}}$, we may replace v by vx in (2.21) and obtain $(vx)\theta = t(vx) = (tv)x = (v\theta)x$ so that (2.20) holds. Thus (2.21) sets up a one to one correspondence between the elements of $\mathscr{H}$ and those of $G_{\mathbb{C}}$. Every endomorphism of the G-module $G_{\mathbb{C}}$ is equivalent to the left multiplication by a fixed element of $G_{\mathbb{C}}$.

The correspondence (2.20) is an isomorphism between the

vector spaces $\mathscr{H}$ and $G_{\mathbb{C}}$. For if $v\theta = tv$ and $v\eta = sv$, where $\eta \in \mathscr{H}$, then

$$\begin{aligned} v(a\theta + b\eta) &= av\theta + bv\eta \\ &= atv + bsv \\ &= (at + bs)v \end{aligned}$$

where $a, b \in \mathbb{C}$. Hence it follows that dimension of $\mathscr{H} = dim\ G_{\mathbb{C}} = g$. Now, we examine the centre of $\mathscr{H}$. Let ξ be a typical element of centre of the $\mathscr{H}$.

$$\forall v \in G_{\mathbb{C}},\ \ v\theta\xi = v\xi\theta, \quad \theta \in \mathscr{H} \tag{2.22}$$

Let $v\xi = zv$ for $v \in G_{\mathbb{C}}$. Thus (2.22) is equivalent to

$$ztv = tzv \quad \text{for} \quad v \in G_{\mathbb{C}} \tag{2.23}$$

Since (2.23) for all $v \in G_{\mathbb{C}}$, putting $v = 1$, we get $zt = tz$. Let $t = \sum_{j=1}^{g} \alpha_j x_j$, where $\alpha_1, \alpha_2, \ldots, \alpha_g$ are arbitrary complex numbers. Hence $zt = tz$ is equivalent to the conditions $zx_j = x_j z$ $(j = 1, 2, 3, \ldots, g)$; i.e. $x_j^{-1} z x_j = z$ $(j = 1, 2, 3, \ldots, g)$. Since x_j can be any one of the elements of G, we can say that z lies in the centre of $G_{\mathbb{C}}$ and consequently, ξ lies in the centre of $\mathscr{H}$ if and only if

$$u^{-1} z u = z \quad \text{for} \quad u \in G. \tag{2.24}$$

Let

$$z = \sum_{i=1}^{g} \lambda_j x_j \tag{2.25}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_g$ are suitable complex numbers. Then

from (2.24) we have $z = \sum_{i=1}^{g} \lambda_j u^{-1} x_j u$. Thus conjugate basis elements have the same coefficients in z. Now suppose that the k conjugacy classes of G are given by $C_1 = \{1\}$, $C_2 = \{y_2, p_2^{-1} y_2 p_2, q_2^{-1} y_2 q_2, \dots\}$,$\dots$,$C_k = \{y_k, p_k^{-1} y_k p_k, q_k^{-1} y_k q_k, \dots\}$.

In general, $C_\alpha = \{y_\alpha, p_\alpha^{-1} y_\alpha p_\alpha, q_\alpha^{-1} y_\alpha q_\alpha, \dots\}$. Now we associate with the conjugacy classes of C_α the element $c_\alpha = y_\alpha + p_\alpha^{-1} y_\alpha p_\alpha + q_\alpha^{-1} y_\alpha q_\alpha + \cdots$ which is an element of $G_{\mathbb{C}}$. On gathering the elements of C_α in the expression (2.25) for each α and suitably renaming the coefficients, we can write (2.25) as $z = \gamma_1 c_1 + \gamma_2 c_2 + \cdots + \gamma_k c_k$ where each γ_k is one of the λs.

Conversely, we have $u^{-1} c_\alpha u = c_\alpha$ for $\alpha = 1, 2, \dots, k$ because the element on the LHS consists of the same terms as c_α though possibly in a different order. Thus c_α lies in the centre of $G_{\mathbb{C}}$ and so does $z = \gamma_1 c_1 + \gamma_2 c_2 + \cdots + \gamma_k c_k$. Clearly $c_1, c_2, \dots, c_k$ are linearly independent because their terms are mutually disjoint. Hence we have obtained $k = dim\ centre(G_{\mathbb{C}}) = dim\ centre(\mathscr{H})$. Hence by the theorem related to the endomorphism algebra, we get the representations involve precisely k inequivalent irreducible constituents. Thus we have onbtained $r = k$.

Then $\sum_{i=1}^{k} \left(f^{(i)}\right)^2 = g$. □

2.3 Character table

In a character table, the complete information about the characters of a group G is displayed. A character is constant on each of the conjugacy classes. If $x \in C_\alpha$, we put $\chi(x) = \chi_\alpha$. Thus it is sufficient to record the values χ_α $(1 \leq \alpha \leq k)$, where k is the number of conjugacy classes. Denote the number of elements in C_α by h_α. So we get the class equation

$$h_1 + h_2 + \cdots + h_k = g \tag{2.26}$$

We always take $C_1 = \{1\}$ and $F^{(1)}$ as the trivial representation $F^{(1)}(x) = 1$ $(\forall x \in G)$. The degree of $F^{(i)}$ is denoted by $f^{(i)}$ so that $\chi^{(i)}(1) = f^{(i)}$ for $i = 1, 2, 3, \ldots k$. The body of the character table is a $k \times k$ square matrix where rows correspond to different characters while each column contains the values of all simple characters for a particular conjugacy class.

	G:					
	C_1	C_2	$\cdots$	C_α	$\cdots$	C_k
	h_1	h_2	$\cdots$	h_α	$\cdots$	h_k
$\chi^{(1)}$	1	1	$\cdots$	1	$\cdots$	1
$\chi^{(2)}$	$f^{(2)}$	$\chi_2^{(2)}$	$\cdots$	$\chi_\alpha^{(2)}$	$\cdots$	$\chi_k^{(2)}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
$\chi^{(\alpha)}$	$f^{(\alpha)}$	$\chi_2^{(\alpha)}$	$\cdots$	$\chi_\alpha^{(\alpha)}$	$\cdots$	$\chi_k^{(\alpha)}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
$\chi^{(k)}$	$f^{(k)}$	$\chi_2^{(k)}$	$\cdots$	$\chi_\alpha^{(k)}$	$\cdots$	$\chi_k^{(k)}$

Table 2.1:

Consider the group S_3: $C_1 = \{1\}, C_2 = \{(12), (13), (23)\}$, $C_3 = \{(123), (132)\}$. Thus $h_1 = 1$, $h_2 = 3$, $h_3 = 2$. The number of simple characters of S_3 is 3 and they are:

(i) the trivial character $\chi^{(1)}(x)$

(ii) the alternating character $\chi^{(2)}(x)$

(iii) $\chi^{(3)}(x) = \phi(x)$, where $\phi(x)$ is the character of the 2 dimensional representation $D(x)$ of S_3.

Hence the character table of S_3 is as follows:

S_3 :

	C_1	C_2	C_3
	$h_1 = 1$	$h_2 = 3$	$h_3 = 2$
$\chi^{(1)}$	1	1	1
$\chi^{(2)}$	1	-1	1
$\chi^{(3)}$	2	0	-1

Table 2.2: S_3

Remark 29. $(f^{(1)})^2 + (f^{(2)})^2 + (f^{(3)})^2 = 1^2 + 1^2 + 2^2 = 6$.

Let $A(x)$ be an arbitrary representation of degree m not necessarily irreducible, and let $\phi(x)$ be the character of $A(x)$. Then, $\phi(x) = tr(A(x))$ = sum of the latent roots of $A(x)$. If ϵ is an eigenvalue of $A(x)$, then ϵ^n is an eigenvalue of $(A(x))^n = A(x^n)$ for $n = 0, \pm 1, \pm 2, \ldots$. In a finite group of order g, each element satisfies the equation $x^g = 1$. Thus

ϵ^g is an eigenvalue of $A(x^g) = A(1) = I$. Thus $\epsilon^g = 1$; i.e. ϵ is a g^{th} root of unity and hence $|\epsilon| = 1$. Thus $\epsilon^{-1} = \overline{\epsilon}$, the complex conjugate of ϵ. If the eigenvalues of $A(x)$ are $\epsilon_1(x), \epsilon_2(x), \ldots, \epsilon_m(x)$, then the eigenvalues of $A(x^{-1})$ are $\overline{\epsilon_1(x)}, \overline{\epsilon_2(x)}, \ldots, \overline{\epsilon_m(x)}$. Thus for any character ϕ, since $\phi(x)$ is the sum of the eigenvalues of $A(x)$, we get $\phi(x^{-1}) = \overline{\phi(x)}$ where $\overline{\phi(x)}$ denotes the complex conjugate of $\phi(x)$.
Now consider $\langle \phi, \psi \rangle$. When $x \in C_\alpha$, then $\phi(x) = \phi_\alpha$, $\psi(x) = \psi_\alpha$ and $\psi(x^{-1}) = \overline{\psi_\alpha}$. Thus, for $x \in C_\alpha$, $\langle \phi, \psi \rangle = \frac{1}{g} \sum\limits_{x \in G} \phi(x)\psi(x^{-1})$

$$= \frac{1}{g} \sum_{\alpha=1}^{k} h_\alpha \phi_\alpha \overline{\psi_\alpha}. \tag{2.27}$$

If χ and χ' are simple characters from (2.27), we get

$$\frac{1}{g} \sum_{\alpha=1}^{k} h_\alpha \chi_\alpha \overline{\chi'_\alpha} = \begin{cases} 0, & \text{if } \chi \neq \chi' \\ 1, & \text{if } \chi = \chi' \end{cases}$$

from the character relation of the first kind. Applying this result to all the rows of the character table, we get

$$\frac{1}{g} \sum_{\alpha=1}^{k} h_\alpha \chi_\alpha^{(i)} \overline{\chi_\alpha^{(j)}} = \delta_{ij} \tag{2.28}$$

$(i, j = 1, 2, \ldots, k)$. Introducing the $k \times k$ matrix $U = (u_{i\alpha}) = \chi_\alpha^{(i)} \sqrt{\frac{h_\alpha}{g}}$, equation (2.28) states that the rows of the matrix U form a system of unitary orthogonal k-tuples. Thus U is a unitary matrix; i.e. $U\overline{U'} = I$, where $'$ denotes transposition. If U is a unitary matrix, then so is U'. This means

that the columns of U also form a unitary orthogonal system;
i.e. $\sum_{i=1}^{k} u_{i\alpha}\overline{u}_{i\beta} = \delta_{\alpha\beta}$. In terms of characters:

$$\sum_{i=1}^{k} \chi_{\alpha}^{(i)} \sqrt{\frac{h_\alpha}{g}} \overline{\chi}_{\beta}^{(i)} \overline{\sqrt{\frac{h_\beta}{g}}} = \delta_{\alpha\beta}$$

$$\implies \sum_{i=1}^{k} \chi_{\alpha}^{(i)} \overline{\chi}_{\beta}^{(i)} = \frac{g}{h_\alpha}\delta_{\alpha\beta}.$$

These equations are called character relations of the second kind. When x ranges over the conjugacy class C_α, x^{-1} ranges over the conjugacy class $C_{\alpha'}$, whose elements are in one to one correspondence with those of C_α. We say that the classes C_α and $C_{\alpha'}$ are inverse to each other. If $C_\alpha = C_{\alpha'}$, then C_α is termed self-inverse. For each character $\phi(x)$, $\phi_{\alpha'} = \overline{\phi_\alpha}$.
Thus the character relations of the second kind can also be written as $\sum_{i=1}^{k} \chi_{\alpha}^{(i)} \chi_{\beta'}^{(i)} = \frac{g}{h_\alpha}\delta_{\alpha\beta}$.

2.4 Finite Abelian Groups

When G is a finite abelian group, each element forms a conjugacy class by itself, so that $k = g$;
i.e. $\sum_{i=1}^{g} [f^{(i)}]^2 = g \implies f^{(i)} = 1$ for $i = 1, 2, 3, \ldots, g$.
Hence all absolutely irreducible representations of a finite abelian group are linear and $f^{(i)}(x) = \chi^{(i)}(x)$ so that $\chi^{(i)}(xy) = \chi^{(i)}(x)\chi^{(i)}(y)$ for linear characters. First suppose that $G = \mathbb{Z}_n$, a cyclic group of order n. We are using

$e(x) = \exp(x)$. Let $\epsilon_r = e(\frac{2\pi ir}{n})$ be any n^{th} root of unity, where r is a fixed integer such that $0 \leq r \leq n-1$. If $\mathbb{Z}_n = gp\{z\}$, $z^n = 1$, we define for an arbitrary element, $z^s \in \mathbb{Z}_n$, $0 \leq s \leq n-1$, $\chi^{(r)}(z^s) = \epsilon_r^s = e(\frac{2\pi irs}{n})$. Then $\chi^{(r)}$ is a linear character of $\mathbb{Z}_n$, since for z^s and $z^t \in \mathbb{Z}_n$,

$$\begin{aligned}\chi^{(r)}(z^s)\chi^{(r)}(z^t) &= e\left(\frac{2\pi irs}{n}\right)e\left(\frac{2\pi irt}{n}\right)\\ &= e\left(\frac{2\pi ir(s+t)}{n}\right)\\ &= \chi^{(r)}(z^{s+t})\\ &= \chi^{(r)}(z^s z^t)\end{aligned}$$

Distinct values of r yield distinct characters and thus we get 'n' characters. Here we denote the trivial character by $\chi^{(0)}$. Thus the complete character table of $\mathbb{Z}_n$ is summarized by: $\chi^{(r)}(z^s) = e(\frac{2\pi irs}{n}) \quad (0 \leq r, s \leq n-1)$. The orthogonality relation $\langle \chi^{(i)}, \chi^{(j)} \rangle = \delta_{ij}$ follows from the elementary identities for exponential functions.

Example 30. *The group $\mathbb{Z}_3$ consists of the elements $1, z, z^2$ where $z^3 = 1$. Let $\omega = e(\frac{2\pi i}{3})$. Then the character table is as follows:*

$\mathbb{Z}_3$:

	1	z	z^2
$\chi^{(0)}$	1	1	1
$\chi^{(1)}$	1	ω	ω^2
$\chi^{(2)}$	1	ω^2	ω

Table 2.3:

Theorem 31. *Suppose that G is a direct-product of m cyclic groups of orders $n_1, n_2, \ldots, n_m$ respectively and $g = |G| = n_1 n_2 \ldots n_m$. Then a typical element of G can be represented by the m-tuple, $x = (a_1, a_2, \ldots, a_m) \quad (0 \leq a_\mu < n_\mu)$. Composition of elements is carried out by addition of components followed by reduction of the μ^{th} component to its least non-negative remainder modulo n_μ where $\mu = 1, 2, \ldots m$. Corresponding to each m-tuple $r = [r_1, r_2, \ldots, r_m]$ where $0 \leq r_\mu < n_\mu$. There exists a linear character $\chi^{[r]}(x) = e\left(2\pi i \sum_{\mu=1}^{m} \frac{a_\mu r_\mu}{n_\mu}\right)$ of G and all g characters form a multiplicative group G^* which is isomorphic to G by virtue of the correspondence $(a_1, a_2, \ldots, a_m) \leftrightarrow [a_1, a_2, \ldots, a_m]$.*

Proof. Let $G = gp\{z_1\} \times gp\{z_2\} \times \cdots \times gp\{z_m\}$, where z_μ is of order n_μ and $g = n_1 n_2 \cdots n_m$. Then an arbitrary element $x \in G$ is uniquely expressed as $x = z_1^{a_1} z_2^{a_2} \cdots z_m^{a_m}$, $(0 \leq a_\mu < n_\mu,\ \mu = 1, 2, \cdots, m)$. In order to construct the simple character of G, we choose for each μ, an n_μ^{th} root of unity, $\epsilon_\mu = e\left(\frac{2\pi i r_\mu}{n_\mu}\right)$, where r_μ is any integer satisfying

$0 \leq r_\mu < n_\mu$, $\mu = 1, \ldots, m$. Corresponding to each m-tuple $r = [r_1, r_2, \ldots, r_m]$,

$$\chi^{[r]}(x) = e\left(2\pi i \sum_{\mu=1}^{m} \frac{a_\mu r_\mu}{n_\mu}\right) \tag{2.29}$$

If $y = z_1^{b_1} z_2^{b_2} \cdots z_m^{b_m}$ is another element of G, then

$$\begin{aligned}
\chi^{[r]}(x)\chi^{[r]}(y) &= e\left(2\pi i \sum_{\mu=1}^{m} \frac{a_\mu r_\mu}{n_\mu}\right) e\left(2\pi i \sum_{\mu=1}^{m} \frac{b_\mu r_\mu}{n_\mu}\right) \\
&= e\left(2\pi i \sum_{\mu=1}^{m} \frac{(a_\mu + b_\mu) r_\mu}{n_\mu}\right) \\
&= \chi^{[r]}(z_1^{a_1+b_1} z_2^{a_2+b_2} \cdots z_m^{a_m+b_m}) \\
&= \chi^{[r]}(z_1^{a_1} z_2^{a_2} \cdots z_m^{a_m} z_1^{b_1} z_2^{b_2} \cdots z_m^{b_m}) \\
&= \chi^{[r]}(xy).
\end{aligned}$$

Since distinct m-tuples correspond to distinct functions, the complete character table is given by (2.29). Let χ and χ' be two functions such that $\chi(xy) = \chi(x)\chi(y)$ and $\chi'(xy) = \chi'(x)\chi'(y)$. Then the function $\chi\chi'$ defined by $\chi\chi'(x) = \chi(x)\chi'(x)$ satisfies $\chi\chi'(xy) = \chi\chi'(x)\chi\chi'(y)$, for

$$\begin{aligned}
\chi\chi'(xy) &= \chi(xy)\chi'(xy) \\
&= \chi(x)\chi(y)\chi'(x)\chi'(y) \\
&= \chi(x)\chi'(x)\chi(y)\chi'(y) \\
&= \chi\chi'(x)\chi\chi'(y).
\end{aligned}$$

So the g simple characters of the finite abelian group G forms a multiplicative group in this sense, which is called the dual

group of G denoted by G^*. Taking $x = (a_1, a_2, \ldots, a_m)$ and $y = (b_1, b_2, \ldots, b_m)$, then $xy = (a_1 + b_1, a_2 + b_2, \ldots, a_m + b_m)$ provided that $a_\mu + b_\mu$ is replaced by its least non-negative remainder *modulo* n_μ. Similarly, if $\chi^{[r]}$ and $\chi^{[s]}$ are two elements of G^*, then, their product in G^* is given by $\chi^{[r]}\chi^{[s]} = \chi^{[r+s]}$, where $[r+s] = [r_1 + s_1, r_2 + s_2, \ldots, r_m + s_m]$ (with the same rule regarding the reduction of $(r_\mu + s_\mu)$ *modulo* n_μ). Thus the groups G and G^* are isomorphic each being equivalent to the group of multiples with the composition described above. □

Example 32. *Find the character table of the four group* $V = gp\{x\} \times gp\{y\}$, *where* $x^2 = y^2 = 1$, $xy = yx$.

Solution. $n_1 = 2$, $n_2 = 2$. *Then* $g = n_1 n_2 = 4$. $|V| = 4$; *i.e.* $V = \{1, x, y, xy\}$. $1 = x^0y^0$, $x = x^1y^0$, $y = x^0y^1$, $xy = x^1y^1$.

$$\begin{aligned}\chi^{[0,0]}(1) &= e\left(2\pi i \sum_{i=1}^{2} \frac{a_\mu r_\mu}{2}\right) \\ &= e\left(2\pi i \frac{(0 \times 0 + 0 \times 0)}{2}\right) \\ &= e(\pi i 0) = 1\end{aligned}$$

Similarly $\chi^{[0,0]}(x) = 1$, $\chi^{[0,0]}(y) = 1$, $\chi^{[0,0]}(xy) = 1$, $\chi^{[0,0]}(1) = 1$, $\chi^{[0,1]}(1) = 1$, $\chi^{[1,0]}(1) = 1$, $\chi^{[1,1]}(1) = 1$. *Now,*

$$\chi^{[0,1]}(x) = e\left(2\pi i\frac{(1\times 0+0\times 1)}{2}\right) = e(0) = 1$$
$$\chi^{[0,1]}(y) = e\left(2\pi i\frac{(0\times 0+1\times 1)}{2}\right) = e(\pi i) = -1$$
$$\chi^{[0,1]}(xy) = e\left(2\pi i\frac{(1\times 0+1\times 1)}{2}\right) = e(\pi i) = -1$$
$$\chi^{[1,0]}(x) = e\left(2\pi i\frac{(1\times 1+0\times 0)}{2}\right) = e(\pi i) = -1$$
$$\chi^{[1,0]}(y) = e\left(2\pi i\frac{(0\times 1+1\times 0)}{2}\right) = e(0) = 1$$
$$\chi^{[1,0]}(xy) = e\left(2\pi i\frac{(1\times 1+1\times 0)}{2}\right) = e(\pi i) = -1$$
$$\chi^{[1,1]}(x) = e\left(2\pi i\frac{(1\times 1+0\times 1)}{2}\right) = e(\pi i) = -1$$
$$\chi^{[1,1]}(y) = e\left(2\pi i\frac{(0\times 1+1\times 1)}{2}\right) = e(0) = -1$$
$$\chi^{[1,1]}(xy) = e\left(2\pi i\frac{(1\times 1+1\times 1)}{2}\right) = e(2\pi i) = 1$$

$\chi^{[0,0]} = \chi^{(1)}$, $\chi^{[1,0]} = \chi^{(2)}$, $\chi^{[0,1]} = \chi^{(3)}$, $\chi^{[1,1]} = \chi^{(4)}$, *and the character table is*

	V :			
	1	x	y	xy
$\chi^{(1)}$	1	1	1	1
$\chi^{(2)}$	1	-1	1	-1
$\chi^{(3)}$	1	1	-1	-1
$\chi^{(4)}$	1	-1	-1	1

Table 2.4:

Theorem 33. *Suppose that the $m \times m$ matrices over $\mathbb{C}$,*

$A_1(=I), A_2, \ldots, A_g$ form an abelian group under multiplication. Then these matrices can be simultaneously diagonalized; i.e. there exists a non-singular matrix T such that $T^{-1}AT = diag(\alpha_1^{(i)}, \alpha_2^{(i)}, \ldots, \alpha_m^{(i)}) \quad (i = 1, 2, \cdots, g)$.

Proof. A matrix group may be regarded as its own representation and is therefore equivalent to diagonally arranged blocks of irreducible representations. In the case of an abelian group, all irreducible representations are linear which amount to the possibility of simultaneous diagonalization. □

Corollary 34. *Every periodic matrix, i.e. a matrix satisfying $A^p = I$ for some positive integer p can be diagonalized.*

Proof. Assume that $p > 1$. Then the matrices $I, A, A^2, \ldots A^{p-1}$ form a cyclic group of order p. Hence there exists a non-singular matrix T such that $T^{-1}AT$ is a diagonal matrix. □

2.5 The Lifting Process

Consider a group G where $g = |G|$, and let

$$N : u_1(=1), u_2, \ldots, u_n$$

be a normal subgroup of G of order n. The elements of the quotient group G/N are the cosets Nx $(x \in G)$ of which exactly $\frac{g}{n}$ are distinct. The law of composition in G/N is

$NxNy = Nxy$ and the identity is N. Also $Nx = Ny$ if and only if $y = ux$ for $u \in N$. In particular, $Nu = N$ if $u \in N$. Suppose that $A_0(Nx)$ is a representation of G/N having degree m. Then, $A_0(Nx)A_0(Ny) = A_0(Nxy)$ and $A_0(N) = I_m$. We let $\phi_0(Nx) = tr(A_0(Nx))$, the character of $A_0(Nx)$.

Theorem 35. *Let N be a normal subgroup of G and suppose that $(A_0(Nx))$ is a representation of degree m of the group G/N. Then $A(x) = (A_0(Nx))$ defines a representation of G lifted from G/N. If $\phi_0(Nx)$ is the character of $A_0(Nx)$, then $\phi(x) = \phi_0(Nx)$ is the lifted character of $A(x)$. Also, if $u \in N$, $A(u) = I_m$ and $\phi(u) = m = \phi(1)$. Then the lifting operation preserves irreducibility.*

Proof. $A(x)A(y) = A_0(Nx)A_0(Ny) = A_0(Nxy) = A(xy)$. Hence A is a representation of G. Let $u \in N$. Then $A(u) = A_0(Nu) = A_0(N) = I_m$. Also, $\phi(u) = \phi_0(Nu) = tr(A_0(Nu)) = tr(I_m) = m$. Since $A(x)$ consists of the same matrices as $A_0(Nx)$, the lifting operation preserves irreducibility. □

Theorem 36. *If A is a representation of G with character ϕ, then those elements u which satisfy $\phi(u) = \phi(1)$ form a normal subgroup M which is the kernel of A. Then $A_0(Mx) = A(x)$ is a well-defined representation of G/M and we may regard A as having being lifted from A_0.*

Proof. Let $A(x)$ be a representation of degree m, and let M be the kernel of A.

If $u \in M$, then $A(u) = I_m \implies \phi(u) = m = \phi(1)$. If $u \in G$ satisfies $\phi(u) = \phi(1) = m$, then since $A(u)$ is a periodic matrix, we have $[A(u)]^p = I$.
$(u \in G \implies u^p = 1 \implies A(u^p) = I \implies [A(u)]^p = I)$.
Then $I, [A(u)], [A(u)]^2, \ldots, [A(u)]^{p-1}$ form a cyclic group of order p. So, $A(u)$ is diagonalizable. Also

$$A(u) \sim diag(\epsilon_1, \epsilon_2, \cdots, \epsilon_m) \tag{2.30}$$

where ϵ_i is a p^{th} root of unity. (for the cyclic group generated by $\mathbb{Z}$ where $z^n = 1$, the characters are the n^{th} roots of unity. Since there is an isomorphism between the group and the characters of the group in the case of an abelian group and the representations are all linear, the representation coincide with the character). Taking the trace on both sides of (2.30),
$m = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_m \implies m \leq |\epsilon_1| + |\epsilon_2| + \cdots + |\epsilon_m| = m$.
The equality $|\epsilon_1 + \epsilon_2 + \cdots + \epsilon_m| = |\epsilon_1| + |\epsilon_2| + \cdots + |\epsilon_m|$ holds only if $\epsilon_1 = \epsilon_2 = \cdots = \epsilon_m = \epsilon$ (see Figure2.1). Therefore, $m\epsilon = m$ implies $\epsilon = 1$. This means that $A(u) = I_m$; i.e. $u \in M$, the kernel of A. Therefore, M is a normal subgroup. To show that $A_0(Mx) = A(x)$ is a well-defined representation of G/M.
Let $Mx = My$. Then $y = ux$ for $u \in M$. Therefore, $A_0(My) = A(y) = A(ux) = A(u)A(x) = I_m A_0(Mx) = A_0(Mx)$. So we may regard A as having being lifted from A_0.

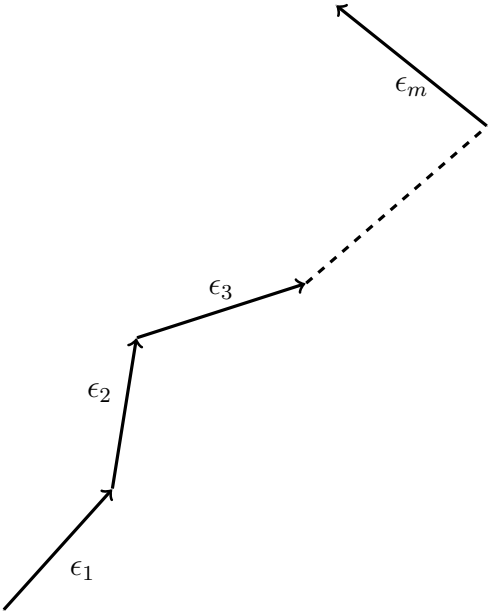

Figure 2.1:

□

Example 37. *The alternating group A_4 is of order* 12 *and consists of the following elements:* $(1), (123), (132), (124), (142), (134), (143), (234), (243), (12)(34), (13)(24), (14)(23)$. *First we must split A_4 into conjugacy classes in the full symmetric group S_4. Two permutations are conjugates if and only if they possesses the same cycle pattern. The conjugacy classes of A_4 are as follows:*
$C_1 = \{1\}$, $C_2 = \{(123), (142), (134), (243)\}$,

$C_3 = \{(132), (124), (143), (234)\}$,
$C_4 = \{(12)(34), (13)(24), (14)(23)\}$.
Since there are four conjugacy classes, A_4 possesses four irreducible representations. Consider $C_1 \cup C_4 = V$. Since $|V| = 4 = 2^2$, it is abelian and hence V is a normal subgroup of G. The quotient group A_4/V is of order 3 and is therefore isomorphic to $\mathbb{Z}_3$. According to lifting process we can lift the three characters of $\mathbb{Z}_3$, all of which are linear, to obtain three characters $\chi^{(1)}, \chi^{(2)}, \chi^{(3)}$ of A_4, the kernel being V. It remains to find $\chi^{(4)}$.
Since $(f^{(1)})^2 + (f^{(2)})^2 + (f^{(3)})^2 + (f^{(4)})^2 = 12$, and $f^{(1)} = f^{(2)} = f^{(3)} = 1$, we deduce that $f^{(4)} = 3$. Denote the values of $\chi^{(4)}$ for the classes C_2, C_3, C_4 by α, β, γ respectively. Thus the character table of A_4 is of the form:

A_4 :

	C_1	C_2	C_3	C_4
h	1	4	4	3
$\chi^{(1)}$	1	1	1	1
$\chi^{(2)}$	1	ω	ω^2	1
$\chi^{(3)}$	1	ω^2	ω	1
$\chi^{(4)}$	3	α	β	γ

Table 2.5:

The values of α, β and γ are readily derived from the orthogonality of the columns (applied to 1st and 2nd columns), the character relations of the second kind. Thus it is found

that $\alpha = \beta = 0$, *because* $1 + \omega + \omega^2 = 0$ *and* $3\gamma + 3 = 0$, *whence* $\gamma = -1$. $1 + \omega + \omega^2 + 3\alpha = 0$, $1 + \omega + \omega^2 + 3\beta = 0$. *Hence the complete character table of* A_4 *is given by:*

A_4 :

	C_1	C_2	C_3	C_4
h	1	4	4	3
$\chi^{(1)}$	1	1	1	1
$\chi^{(2)}$	1	ω	ω^2	1
$\chi^{(3)}$	1	ω^2	ω	1
$\chi^{(4)}$	3	0	0	-1

Table 2.6:

Example 38. *The dihedral group* D_4 *of order* 8 *is defined by* $a^4 = b^2 = 1$, $b^{-1}ab = a^3$.
Centre of $D_4 = \{z \in D_4 : u^{-1}zu = z, u \in D_4\}$.
Result: *Let* $n \in \mathbb{N}$ *such that* $n \geq 3$. *Let* D_n *be the dihedral group of order* $2n$, *given by:*
$D_n = \langle a, b : a^n = b^2 = e, b^{-1}ab = a^{-1} \rangle$. *Then*
$Z(D_n) = \begin{cases} e & : n \text{ odd}, \\ \{e, a^{\frac{n}{2}}\} & : n \text{ even}. \end{cases}$
Therefore, centre of $D_4 = \{1, a^2\}$. *Centre form conjugacy classes by themselves, the other conjugacy classes are* $\{a, a^3\}$, $\{b, a^2b\}$, $\{ab, a^3b\}$. *Therefore the conjugacy classes of* D_4 *are* $\{1\}$, $\{a^2\}$, $\{a, a^3\}$, $\{b, a^2b\}$, $\{ab, a^3b\}$. *Since there are five conjugacy classes in all, the group* D_4 *possesses five irreducible representation. Since centre of a group is a normal*

subgroup, we have the subgroup $N = \{1\} \cup \{a^2\}$ *is normal in* D_4*. Consider the quotient group* D_4/N*. It consists of the elements* $N = \{1, a^2\}$*;* $Na = \{a, a^3\}$*;* $Nb = \{b, a^2b\}$*;* $Nab = \{ab, a^3b\}$*. Here we have* $(Na)^2 = Na^2 = N$ *(since* $a^2 \in N$*),* $(Nb)^2 = N$ *as* $b^2 = 1$*,* $(Nab)^2 = Na^2b^2 = Nb^2 = N$*; i.e. square of every element in* D_4/N *belongs to* N*. Hence every non identity element has order* 2*. So* D_4/N *is isomorphic to the four group* $V = gp\{x\} \times gp\{y\}$ *whose character table is given by:*

V :

	1	x	y	xy
$\chi^{(1)}$	1	1	1	1
$\chi^{(2)}$	1	-1	1	-1
$\chi^{(3)}$	1	1	-1	-1
$\chi^{(4)}$	1	-1	-1	1

Table 2.7:

The four characters of D_4/N *are lifted to yield linear character* $\chi^{(1)}, \chi^{(2)}, \chi^{(3)}, \chi^{(4)}$ *of* D_4*. It remains to determine* $\chi^{(5)}$*. Since* $f^{(1)} = f^{(2)} = f^{(3)} = f^{(4)} = 1$ *and* $\sum_{i=1}^{5} [f^{(i)}]^2 = 8$*, it follows that* $f^{(5)} = 2$*. The other values of* $\chi^{(5)}$ *are deduced from the fact that the columns of the character table are mutually orthogonal.*

Consider the table :

D_4 :

	1	1	2	2	2
$\chi^{(1)}$	1	1	1	1	1
$\chi^{(2)}$	1	1	1	-1	-1
$\chi^{(3)}$	1	1	-1	1	-1
$\chi^{(4)}$	1	1	-1	-1	1
$\chi^{(5)}$	2	x	y	z	w

Table 2.8:

Since columns (1) *and* (2) *are orthogonal, we have*

$$(1 \times 1) + (1 \times 1) + (1 \times 1) + (1 \times 1) + (2 \times x) = 0$$
$$\implies 1 + 1 + 1 + 1 + 2x = 0$$
$$\implies x = -2.$$

Since columns (1) *and* (3) *are orthogonal, we have*

$$(1 \times 1) + (1 \times 1) + (1 \times -1) + (1 \times -1) + (2 \times y) = 0$$
$$\implies 1 + 1 - 1 - 1 + 2y = 0$$
$$\implies y = 0.$$

Since columns (1) *and* (4) *are orthogonal, we have*

$$(1 \times 1) + (1 \times -1) + (1 \times 1) + (1 \times -1) + (2 \times z) = 0$$
$$\implies 1 - 1 + 1 - 1 + 2z = 0$$
$$\implies z = 0.$$

Since columns (1) *and* (5) *are orthogonal, we have*

$$(1 \times 1) + (1 \times -1) + (1 \times -1) + (1 \times 1) + (2 \times w) = 0$$
$$\implies 1 - 1 - 1 + 1 + 2w = 0$$
$$\implies w = 0.$$

Thus the final character table for D_4 is:

D_4 :

	1	1	2	2	2
$\chi^{(1)}$	1	1	1	1	1
$\chi^{(2)}$	1	1	1	-1	-1
$\chi^{(3)}$	1	1	-1	1	-1
$\chi^{(4)}$	1	1	-1	-1	1
$\chi^{(5)}$	2	-2	0	0	0

Table 2.9:

Example 39. *The quaternion group Q of order 8 is defined by $a^4 = 1$, $a^2 = b^2$, $b^{-1}ab = a^3$. (It is not isomorphic to D_4.) The group $N = \{1, a^2\}$ is a normal subgroup of Q, and in fact is its centre. Also the group Q/N consists of the cosets N, Na, Nb, Nab and is isomorphic to the four-group. Hence the same arguments apply for the construction of the character table of Q as in the case of D_4. Thus these two groups have the same character tables. This is a somewhat disappointing discovery, as it dashes all hope of using the character table to discriminate between non-isomorphic groups.*

2.6 Linear Characters

The problem of finding all the linear characters of G is related to the commutator group or derived group $G' = gp\{x^{-1}y^{-1}xy : x, y \in G\}$. G' is normal in G and G/G' is abelian. Also, G' is the least subgroup with these properties.

Theorem 40. *The number of linear characters of $G = |G/G'|$. There is a one to one correspondence between the simple characters λ_0 of G/G' which are necessarily linear and the linear characters of G given by $\lambda(x) = \lambda_0(G'x)$ for $x \in G$. All linear characters of G take the value unity on G'.*

Proof. Since G' is normal in G, G/G' is abelian and hence all its simple characters are linear. Let $|G/G'| = l$. Let

$$\lambda_0^{(1)}, \ldots, \lambda_0^{(l)} \tag{2.31}$$

be the simple characters of G/G'. Then we obtain l characters of G using the lifting process namely

$$\lambda^{(i)}(x) = \lambda_0^{(i)}(G'x) \tag{2.32}$$

for $i = 1, 2, 3, \ldots, l$. Conversely, we will show that all the linear characters of G are obtained in this way. Let $\mu(x)$ be a linear character of G. For any commutator $a^{-1}b^{-1}ab$, $\mu(a^{-1}b^{-1}ab) = \mu(a)^{-1}\mu(b)^{-1}\mu(a)\mu(b) = 1$. Since the commutator generates G', we get $\mu(u) = 1$ for all $u \in G'$. Thus, G' lies in the kernel of every linear representation of G; i.e. linear characters of G take the value 1 on G'. Now, we define

a representation $\mu_0(G/G')$ by

$$\mu_0(G'x) = \mu(x) \tag{2.33}$$

If $G'x = G'y$, then $y = ux$ for $u \in G'$. Therefore, $\mu(y) = \mu(ux) = \mu(u)\mu(x) = 1\mu(x) = \mu(x)$. So, we deduce that μ_0 must be one of the representations of G/G' listed in (2.31), and it follows from (2.33) that μ is one of the representations defined in (2.32). □

Note. *If $A(x)$ is any representation of G, irreducible or not and $\lambda(x)$ is any linear character, then $B(x) = \lambda(x)A(x)$ is also a representation. $B(xy) = \lambda(xy)A(xy) = \lambda(x)\lambda(y)A(x)A(y) = \lambda(x)A(x)\lambda(y)A(y) = B(x)B(y)$. If $A(x)$ has character $\phi(x)$, then $B(x)$ has character $\lambda(x)\phi(x)$.*

Proposition 41. *If $\lambda(x)$ is a linear character, then together with every irreducible character $\chi(x)$, the character table also contains $\lambda(x)\chi(x)$.*

Proof. Consider

$$\begin{aligned}
\langle \lambda\chi, \lambda\chi \rangle &= \frac{1}{g} \sum_{x \in G} \lambda(x)\chi(x)\lambda(x^{-1})\chi(x^{-1}) \\
&= \frac{1}{g} \sum_{x \in G} \lambda(x)\lambda(x^{-1})\chi(x)\chi(x^{-1}) \\
&= \frac{1}{g} \sum_{x \in G} \lambda(1)\chi(x)\chi(x^{-1}) \\
&= \frac{1}{g} \sum_{x \in G} \chi(x)\chi(x^{-1}) \\
&= \langle \chi, \chi \rangle = 1,
\end{aligned}$$

since χ is a simple character. □

2.7 Exercises

1. Let $A = (a_{ij}(x))$ be an absolutely irreducible representation of a finite group G. Prove that $\sum_{y\in G} a_{ip}(y^{-1})a_{ps}(yx) = \frac{g}{f}a_{is}(x)$ for $x \in G$, where $g = |G|$, $f = deg(A)$.

2. Let A be an absolutely irreducible representation of degree f of a finite group G. If the $f \times f$ matrix K has the property that $tr(KA(x)) = 0$, $\forall x \in G$, then $K = 0$.

3. Show that the permutation representation of S_3 induced by the subgroup $H = \{1, (12)\}$ is isomorphic to S_3.

4. Show that in a group G of odd order no element other than 1 is conjugate to its inverse. Prove that if $u \in G$ and $u \neq 1$, there exists at least one simple character χ such that $\chi(u)$ is not real.

Solutions to Exercises

1. We have
$$\sum_{y\in G} a_{ip}(y^{-1})a_{pj}(y) = \frac{g}{f}\delta_{ij}.$$
Multiplying the above equation by $a_{js}(x)$ and summing over j, $\sum_{y\in G} a_{ip}(y^{-1}) \sum_{j=1}^{f} a_{pj}(y)a_{js}(x) = \frac{g}{f}a_{is}(x)$.

$\sum_{j=1}^{f} a_{pj}(y)a_{js}(x)$ is the $(p,s)^{th}$ entry of $A(y)A(x) = A(yx)$ which is $a_{ps}(yx)$.

Thus we have proved that $\sum_{y\in G} a_{ip}(y^{-1})a_{ps}(yx) = \frac{g}{f}a_{is}(x)$.

2. Let $K = (k_{ij})$ and $A(x) = (a_{ij}(x))$. The diagonal elements of KA are given by the $(i,i)^{th}$ entries for $i = 1, 2, \ldots, f$. So the $(i,i)^{th}$ entry of $KA(x)$ is obtained by multiplying i^{th} row of K by i^{th} column of $A(x)$. So the $(i,i)^{th}$ entry is $\sum_{r=1}^{f} k_{ir}a_{ri}(x)$.
Therefore, $tr(KA(x)) = 0$ $(\forall x \in G)$ implies

$$\sum_{i,r=1} k_{ir}a_{ri}(x) = 0.$$

Choose p and q arbitrarily and multiply the above equation by $a_{pq}(x^{-1})$,

$$k_{11}a_{11}(x)a_{pq}(x^{-1}) + \cdots + k_{1f}a_{f1}(x)a_{pq}(x^{-1}) + \cdots$$
$$+ k_{f1}a_{1f}(x)a_{pq}(x^{-1}) + \cdots + k_{ff}a_{ff}(x)a_{pq}(x^{-1}) = 0.$$

In particular if $p = 2$, $q = 1$, then since $\langle a_{ij}, a_{pq}\rangle = \frac{1}{f}\delta_{iq}\delta_{jp}$, $k_{21}a_{12}(x)a_{21}(x^{-1}) = \frac{k_{21}}{f}$. So in general, $k_{ij}a_{ji}(x)a_{pq}(x^{-1}) = \frac{k_{ij}}{f}$ when $j = q$, $i = p$. So $k_{ij} = 0$ for all i, j.

3. Since $[S_3 : H] = 3$, the representation $\sigma(x) : Ht_i \to Ht_ix$ $(i = 1, 2, 3)$ certainly maps S_3 into S_3. It

remains to verify that the kernel of σ is the trivial group. By Exercise 1 of chapter 1, kernel of σ is $\bigcap_{i=1}^{3} t_i^{-1} H t_i$. Take $t_1 = 1, t_2 = (13), t_3 = (23)$. Then $Ht_1 \cup Ht_2 \cup Ht_3 = S_3$. $t_2^{-1} H t_2 = (13)H(13) = (13)\{(13), (12)(13)\} = \{1, (231)\}$
$t_3^{-1} H t_3 = (23)H(23) = (23)\{(23), (12)(23)\} = \{1, (31)\}$
Thus $\bigcap_{i=1}^{3} t_i^{-1} H t_i = \{1\}$ and hence the result.

4. There are no elements of order 2 in G. Hence if $u \neq 1$, then $u \neq u^{-1}$. Also, u and u^{-1} cannot be conjugate. For suppose that $t^{-l} u t = u^{-1}$; then $t^{-l} t^{-l} u t t = t^{-l} u^{-1} t$; i.e. $t^{-2} u t^2 = u$, which implies that t^2 would belong to the centraliser $C(u)$. As the order of t is necessarily odd, say $t^{2r+1} = 1$, so that $t = (t^2)^{-r}$, which would imply that $t \in C(u)$; i.e. $t^{-l} u t = u$ which gives $u^{-1} = u$, which is a contradiction. Thus if $u \in C_\alpha$, then $u^{-1} \in C'_\alpha$, where $\alpha \neq \alpha'$. Applying the character relations of the second kind to α and α' we get $\Sigma_{i=1}^{k} (\chi_\alpha^{(i)})^2 = 0$. Thus not all the numbers $\chi_\alpha^{(i)}$ can be real.

Chapter 3

Induced Characters

3.1 Induced representations

Let G be a group of order g and let H be a subgroup of order h and index $n = \frac{g}{h}$. Suppose that $B(u)$, $u \in H$ is a representation of H of degree q. Thus $B(u)B(v) = B(uv)$ for $u, v \in H$. We formally extend the matrix function B to the whole of G. Let $B(x) = 0$ if $x \notin H$. This does not make $B(x)$ a representation of G since all representation consists of non-singular matrices.

Let $G = Ht_1 \cup Ht_2 \cup \cdots \cup Ht_n$ be a decomposition of G into right cosets of H; i.e. $t_1, t_2, \cdots, t_n$ is a right transversal of H in G.

For every element x of G, we define a matrix $A(x)$ of degree qn as an $n \times n$ array of blocks each of degree q as follows.

$$A(x) = \begin{bmatrix} B(t_1xt_1^{-1}) & B(t_1xt_2^{-1}) & \cdots & B(t_1xt_n^{-1}) \\ B(t_2xt_1^{-1}) & B(t_2xt_2^{-1}) & \cdots & B(t_2xt_n^{-1}) \\ \vdots & \vdots & & \vdots \\ B(t_nxt_1^{-1}) & B(t_nxt_2^{-1}) & \cdots & B(t_nxt_n^{-1}) \end{bmatrix}$$

The fundamental discovery of Frobenius was the fact that $A(x)$ is a representation of G; i.e.

$$A(x)A(y) = A(xy) \text{ for } x, y \in G \tag{3.1}$$

Comparing the $(i,j)^{th}$ block on both sides (3.1) is equivalent to

$$B(t_ixyt_j^{-1}) = \sum_{r=1}^{n} B(t_ixt_r^{-1})B(t_ryt_j^{-1}) \tag{3.2}$$

Two cases arise according to the nature of the LHS of (3.2)

(i) Suppose that $t_ixyt_j^{-1} \notin H$. Therefore. LHS of (3.2) is the 0 matrix. We claim that for each r $(1 \leq r \leq n)$, either $t_ixt_r^{-1} \notin H$ or $t_ryt_j^{-1} \notin H$; for otherwise $t_ixt_r^{-1}t_ryt_j^{-1} \in H$ which gives $t_ixyt_j^{-1} \in H$ which contradicts our present hypothesis. Hence each term on Right Hand Side of (3.2) is 0 and the two sides agree.

(ii) Suppose that $v = t_ixyt_j^{-1} \in H$. The element t_ix belongs to exactly one right coset, say Ht_s. Therefore $t_ix \in Ht_s$ so that $u = t_ixt_s^{-1} \in H$. It follows that the sum of the RHS of (3.2) reduces to the single term in which $r = s$. This term is surely non-zero because $t_syt_j^{-1} = u^{-1}v \in H$ and hence $B(u)B(u^{-1}v) = B(v)$ and (3.2) is equivalent to $B(u)B(u^{-1}v) = B(v)$ which

is true since B is a representation of H. Thus, we have shown that $A(x)A(y) = A(xy)$ and hence $A(x)$ is a representation of G.

Also we have shown that for a fixed value of i, exactly one block in the matrix representation $A(x)$ is non-zero. Similarly, for a fixed value of j, precisely one block is non-zero. Thus $A(x)$ can be described as a generalization of a permutation matrix, the units in such a matrix having been replaced by blocks B. The representation $A(x)$ of G is said to have been induced by the representation $B(u)$ of H. Let $\phi(u)$ be the character of $B(u)$. For $x \notin H$, $\phi(x) = 0$. In conformity with the extension $B(x) = 0$ for $x \notin H$, we define $\phi(x) = 0$ for $x \notin H$. The character of $A(x)$, which is called the induced character of ϕ, will be denoted by ϕ^G. Thus $\phi^G(x) = tr(A(x)) = \sum_{i=1}^{n} \phi(t_i x t_i^{-1})$. In the construction of $A(x)$ we employed a particular transversal, but we shall now demonstrate that this choice does not materially affect the result. For $s_1, s_2, \ldots, s_n$ is another transversal if and only if $s_i = u_i t_i$ $(i = 1, 2, \ldots, n)$ where $u_1, u_2, \ldots, u_n$ are suitable elements of H. If $s_i x s_i^{-1} = u_i t_i x t_i^{-1} u_i^{-1}$ lies in H, so does $t_i x t_i^{-1}$ and since these elements are conjugates in H,

$$\phi(s_i x s_i^{-1}) = \phi(t_i x t_i^{-1}). \tag{3.3}$$

But if $s_i x s_i^{-1} \notin H$, then neither does $t_i x t_i^{-1}$ and (3.3) is trivially satisfied because both sides are 0. Now summing both sides of (3.3) over i, the RHS is

$$\phi^G(x) = \sum_{i=1}^{n} \phi(s_i x s_i^{-1}). \tag{3.4}$$

Let u be an arbitrary element of H and choose the transversal $s_i = ut_i$ $(i = 1, 2, \cdots, n)$. Then $\phi^G(x) = \sum_{i=1}^{n} \phi(ut_i x t_i^{-1} u^{-1})$. Summing over u in H, note that the left-hand side is multiplied by h, while the products ut_i, run over G exactly once. Hence $h\phi^G(x) = \sum_{y \in G} \phi(yxy^{-1})$. That is,

$$\phi^G(x) = \frac{1}{h} \sum_{y \in G} \phi(yxy^{-1}). \tag{3.5}$$

Next we consider elements of a particular conjugacy class C_α . Thus if $x \in C_\alpha$, $\phi_\alpha^G = \phi^G(x)$. There are h_α distinct conjugates of x; i.e. $h_\alpha = |C_\alpha|$. As y runs through G, the element yxy^{-1} is always conjugate with x in G, but these elements are not all distinct. We have

$$y_1 x y_1^{-1} = y_2 x y_2^{-1} \tag{3.6}$$

if and only if $y_1^{-1} y_2$ commutes with x.

Definition 42. *The set of all elements of G that commute with x is called the* ***centraliser*** *of x in G, a subgroup of G denoted by Z_x.*

Hence (3.6) is equivalent with $y_1^{-1} y_2 \in Z_x$; i.e. $y_1 Z_x = y_2 Z_x$. Thus we have shown that there is a one-to one correspondence between the distinct conjugates of x and the left cosets of its centraliser in G. Therefore $h_\alpha = [G : Z_x]$, the index of Z_x in G when y runs through G, the elements yxy^{-1}

cover C_α exactly $\frac{g}{h_\alpha}(= |Z_x|)$ times. So (3.5) becomes

$$\begin{aligned}\phi_\alpha^G &= \frac{g}{hh_\alpha}\sum_{z\in C_\alpha}\phi(z)\\ &= \frac{n}{h_\alpha}\sum_{z\in C_\alpha}\phi(z),\end{aligned}$$

where $n = \frac{g}{h}$. Since $\phi(z) = 0$ for $z \notin H$, the above expression may be written as $\phi_\alpha^G = \frac{n}{h_\alpha}\sum\limits_w \phi(w)$ for $w \in C_\alpha \cap H$.

We can see that the permutation representation

$$\sigma(x) = \begin{pmatrix} Ht_1 & Ht_2 & \cdots & Ht_n \\ Ht_1x & Ht_2x & \cdots & Ht_nx \end{pmatrix}$$

of G is a special case of the induced representation which corresponds to the trivial representation of H. Let $\nu(x)$ be the character of $\sigma(x)$, viewed as a representation by permutation matrices. Then $\nu(x)$ is the number of objects fixed under the action of x. $Ht_ix = Ht_i$ if and only if $t_ixt_i^{-1} \in H$. Now denoting the trivial character of H, or its extension to G, by ξ, we have $\xi(u) = 1$, $u \in H$ and $\xi(x) = 0$, $x \notin H$. Hence we can express $\nu(x)$ by the formula $\nu(x) = \sum\limits_{i=1}^{n}\xi(t_ixt_i^{-1})$. Therefore we get $\nu = \xi^G$.

Proposition 43. *If G has a permutation representation $\pi(x)$ of degree greater than unity, then the function $\phi(x) =$ (number of fixed points of $\pi(x)$) $- 1$ is a character of G, possibly compound.*

Proof. The permutation character say ψ is compound when $n > 1$. We have

$$\begin{aligned} c_1 &= \langle \psi, \xi^{(1)} \rangle \\ &= \frac{1}{g} \sum_{\alpha=1}^{k} h_\alpha \psi_\alpha \end{aligned}$$

and it follows that the Fourier expansion of ψ is $\psi = c_1\chi^{(1)} + c_2\chi^{(2)} + \cdots$ where c_1 is a positive integer. Then, $\psi - \chi^{(1)} = (c_1 - 1)\chi^{(1)} + c_2\chi^{(2)} + \cdots$ is also a character. □

3.2 The reciprocity law

The process of induction raises a character of a subgroup H into a character of the full group G. There is a trivial operation that works in the opposite direction. If $A(x)$ is a representation of G, then by restricting x to elements of H we obtain a representation $A(u)$ of H, because the equation $A(u)A(v) = A(uv)$ holds for all $u, v \in H$. If ϕ is the character of A, we denote the restricted(or deduced) character by ϕ_H, that is $\phi_H(u) = \phi(u)$ for $u \in H$. In general the process of restriction does not preserve irreducibility. Thus it may happen that, while $A(x)$ is an irreducible representation of G, the set of matrices $A(u)$, $u \in H$ is reducible over the ground field. When dealing with inner products we must now indicate which group is involved. So we use the notation $\langle \sigma, \rho \rangle_G = \frac{1}{g} \sum_{x \in G} \sigma(x)\overline{\rho}(x)$ and analogously $\langle \sigma, \rho \rangle_H = \frac{1}{h} \sum_{u \in H} \sigma(u)\overline{\rho}(u)$. The process of induction and restriction are linked by a result, which furnishes us with a

useful tool for handling group characters.

Theorem 44. *(Reciprocity Theorem of Frobenius)*
Let H be a subgroup of G. If ψ and ϕ are characters of H and G respectively, then $\langle \psi^G, \phi \rangle_G = \langle \psi, \phi_H \rangle_H$

Proof. Applying equation (3.5) to ψ, we have that $\psi^G(x) = \frac{1}{h} \sum_{y \in G} \phi(yxy^{-1})$ where $\psi(z) = 0$. If $z \notin H$ then,

$$\begin{aligned}
\langle \psi^G, \phi \rangle_G &= \frac{1}{g} \sum_{x \in G} \psi^G(x) \overline{\phi}(x) \\
&= \frac{1}{g} \sum_{x \in G} \frac{1}{h} \sum_{y \in G} \psi(yxy^{-1}) \overline{\phi}(x) \\
&= (gh)^{-1} \sum_{x \in G} \sum_{y \in G} \psi(yxy^{-1}) \overline{\phi}(x) \\
&= (gh)^{-1} \sum_{y \in G} \sum_{x \in G} \psi(yxy^{-1}) \overline{\phi}(x)
\end{aligned}$$

For fixed y, let $z = yxy^{-1}$; i.e. $x = y^{-1}zy$. As x runs through G so does z. Hence we may use z as the summation variable inner sum and obtain that $\langle \psi^G, \phi \rangle_G = (gh)^{-1} \sum_{y \in G} \sum_{z \in G} \psi(z) \overline{\phi}(y^{-1}zy)$. Since ϕ is a class function on G, we have $\phi(y^{-1}zy) = \phi(z)$. The terms are therefore independent of y and the summation with respect to y gives a multiplication by g. Thus, $\langle \psi^G, \phi \rangle_G = (gh)^{-1} g \sum_{z \in G} \psi(z) \overline{\phi}(z) = h^{-1} \sum_{z \in G} \psi(z) \overline{\phi}(z)$. But ψ vanishes when z does not belong to H. Hence the summation may be restricted to H and we get

$$\begin{aligned}\langle \psi^G, \phi \rangle_G &= h^{-1} \sum_{z \in H} \psi(z)\overline{\phi}(z) \\ &= h^{-1} \sum_{z \in H} \psi(z)\overline{\phi}_H(z) \\ &= \langle \psi, \phi_H \rangle_H .\end{aligned}$$

□

3.3 The alternating group A_5

This group which is also the symmetry group of a regular icosahedron, will here be viewed as the set of the sixty even permutations of five objects. Thus A_5 is a subgroup of index two in S_5, the full symmetric group of degree five. Let

$$G = A_5,\ S = S_5.$$

We first split G into conjugacy classes. In S, the conjugacy classes of the even permutations are represented by the cycle patterns

C^* :	1	(12)(34)	(123)	(12345)	comprising of
h^* :	1	15	20	24	elements respectively.

Now, if x, y are in G, are conjugate in S, it does not follow that they are also conjugate in G. Let

$$t^{-1}xt = y \tag{3.7}$$

where t is any permutation and suppose x commutes with some odd permutation s, so that

$$s^{-1}xs = x. \quad (s \notin G) \tag{3.8}$$

Then in addition to (3.7), $(st)^{-1}x(st) = y$, and either t or st is even. Thus x and y remain conjugate if (3.8) holds. But this is different for

$$u = (12345). \tag{3.9}$$

If $C(u)$ is the conjugacy class and Z_u the centralizer of u in G, then

$$|C(u)| = [G : Z_u] \tag{3.10}$$

Similarly, if * refers to the group S, $|C^*(u)| = [S : Z_u^*]$. Now, $|S| = 120$ and $|C^*(u)| = 24$. Hence, $|Z_u^*| = 5$. Since the five elements

$$1, u, u^2, u^3, u^4 \tag{3.11}$$

surely commute with u, and $|Z_u^*| = 5$, we have $Z_u^* = \{1, u, u^2, u^3, u^4\}$. Since all these elements are even, $Z_u^* = Z_u$. We have $|C(u)| = 12$. Thus the class of u in S splits in G. In fact, u, u^2 are conjugate in S but not in G. For $t = (2354)$, an odd permutation, $t^{-1}ut = u^2$. If there were an even permutation w such that $w^{-1}uw = u^2$, then tw^{-1}, an odd permutation would commute with u. This is impossible. Since $t^{-1}ut = u^2$, we get

$$t^{-2}ut^2 = t^{-1}(t^{-1}ut)t = t^{-1}u^2t = u^4$$

Hence, u and u^4 are conjugates in G and so also are u^2 and u^3. Now we describe the conjugacy classes of G;

$$\begin{array}{lccccc} C_\alpha: & 1 & (12)(34) & (123) & u & u^2 \\ h_\alpha: & 1 & 15 & 20 & 12 & 12 \end{array}$$

Thus G has five irreducible representations, and we proceed to find their characters.

(1) Trivial character will be denoted by $\chi^{(1)}$.

(2) Since G is a permutation group, from proposition(43) we have

$$\chi^{(2)} : 4, 0, 1, -1, -1.$$

Since 1 fixes 5 elements, (12)(34) fixes 1 element, (123) fixes 2 elements, u and u^2 fixes 0 elements. Also, $\langle \chi^{(2)}, \chi^{(2)} \rangle_G = \frac{1}{60}(16 + 15 \times 0 + 20 \times 1 + 12 \times 1 + 12 \times 1)$ $= 1$, which shows that $\chi^{(2)}$ is a simple character.

(3) A_4 is a subgroup of A_5. We saw a linear character ϕ of A_4 which is:

$$\begin{array}{lcccc} C'_\alpha: & 1 & (12)(34) & (123) & (123)^2 \\ h'_\alpha: & 1 & 3 & 4 & 4 \\ \phi: & 1 & 1 & \omega & \omega^2 \end{array}$$

where C'_α denotes a class of A_4 and h'_α its size. Let $\omega = \exp(\frac{2\pi i}{3})$. So $\omega + \omega^2 = -1$. Now we construct the induced character ϕ^G from the equation

$$\phi^G_\alpha = \tfrac{n}{h_\alpha} \sum_{w \in C_\alpha \cap A_4} \phi(w).$$

Here, $n = [G : A_4] = 5$. So

$$\phi_1^G = \frac{5}{1}\phi(1) = 5$$
$$\phi_2^G = \frac{5}{15}(3 \times 1) = 1$$
$$\phi_3^G = \frac{5}{20}(4\omega + 4\omega^2) = -1$$
$$\phi_4^G = 0$$
$$\phi_5^G = 0.$$

Since $\langle \phi^G, \phi^G \rangle_G = \frac{1}{60}(25 + 12 + 20) = 1$, ϕ^G is simple and we can denote it by $\chi^{(3)}$.

$$\chi^{(3)} : 5, 1, -1, 0, 0.$$

(4) In order to continue our search for characters we use the cyclic subgroup $U = \{1, u, u^2, u^3, u^4\}$ generated by $u = (12345)$. A non-trivial linear character of U is specified by $\lambda(u) = \epsilon$, where $\epsilon = \exp(\frac{2\pi i}{5})$. Note that ϵ may be replaced by $\epsilon^2, \epsilon^3, \epsilon^4$ since we just need $1 + \epsilon + \epsilon^2 + \epsilon^3 + \epsilon^4 = 0$. We now compute the induced character λ^G. Note that $n = [G : U] = 12$. Now, $C_\alpha \cap U$ for $\alpha = 1, 2, 3, 4, 5$ need to be found. $C_1 \cap U = \{1\}$, $C_2 \cap U = C_3 \cap U = \emptyset$. Since u, u^4 are conjugates in G, $C_4 \cap U = \{u, u^4\}$, and $C_5 \cap U = \{u^2, u^3\}$. Hence $\lambda_1^G = 12$, $\lambda_2^G = \lambda_3^G = 0$, $\lambda_4^G = \epsilon + \epsilon^4$, $\lambda_5^G = \epsilon^2 + \epsilon^3$. We observe that the values of λ^G are real, since $\epsilon + \epsilon^4 = 2\cos(\frac{2\pi}{5})$ and $\epsilon^2 + \epsilon^3 = 2\cos(\frac{4\pi}{5})$. $\langle \lambda^G, \lambda^G \rangle_G = \frac{1}{60}(144 + 12(\epsilon + \epsilon^4)^2 + 12(\epsilon^2 + \epsilon^3)^2) = 3$. Thus λ^G is not a simple character but it is a sum of

three simple characters. Now we enquire whether the Fourier analysis of λ^G involves any of the simple characters we have previously discovered.

$$\langle \lambda^G, \chi^{(1)} \rangle = \frac{1}{60}(12 + 12(\epsilon + \epsilon^4) + 12(\epsilon^2 + \epsilon^3)) = 0$$

$$\langle \lambda^G, \chi^{(2)} \rangle = \frac{1}{60}(12 \times 4 + 12 \times -1(\epsilon + \epsilon^4) + 12 \times -1(\epsilon^2 + \epsilon^3))$$

$$= 1$$

$$\langle \lambda^G, \chi^{(3)} \rangle = \frac{1}{60}(12 \times 5 + 0 + 0) = 1$$

Hence $\lambda^G - \chi^{(2)} - \chi^{(3)}$ is a simple character. Denote it by $\chi^{(4)}$.

$$\chi^{(4)} : 3, -1, 0, -\epsilon^2 - \epsilon^3, -\epsilon - \epsilon^4.$$

(5) Finally, on replacing ϵ by ϵ^2 we obtain the remaining character $\chi^{(5)}$. Other substitutions $\epsilon \to \epsilon^s$ would be permissible but would not lead to new results. Hence the complete character table is:

A_5:

C	1	$(12)(34)$	(123)	(12345)	$(12345)^2$
h	1	15	20	12	12
$\chi^{(1)}$	1	1	1	1	1
$\chi^{(2)}$	4	0	1	-1	-1
$\chi^{(3)}$	5	1	-1	0	0
$\chi^{(4)}$	3	-1	0	$-2\cos\frac{4\pi}{5}$	$-2\cos\frac{2\pi}{5}$
$\chi^{(5)}$	3	-1	0	$-2\cos\frac{2\pi}{5}$	$-2\cos\frac{4\pi}{5}$

Table 3.1:

Conclusion: A_5 has no proper normal subgroup. For if N were such a subgroup, then A_5/N on being lifted to A_5 would be one of the characters included in this table. But a lifted character has the property that $\chi(1) = \chi(y)$ for some $y \in N$, other than 1. This is not the case except for $\chi^{(1)}$, which proves that A_5 is indeed a simple group.

3.4 Normal Subgroups

Consider a group G and a normal subgroup H of order h. Thus if $u \in H$ and $t \in G$, $tut^{-1} \in H$. Suppose that B is a representation of H. $B(u)B(v) = B(uv) \quad (u, v \in H)$. Then we can verify that for each fixed t of G, $B_t(u) = B(tut^{-1})$ form a representation of H.

$$\begin{aligned} B_t(u)B_t(v) &= B(tut^{-1})B(tvt^{-1}) \\ &= B(tut^{-1}tvt^{-1}) \\ &= B(tuvt^{-1}) \\ &= B_t(uv). \end{aligned}$$

If ψ is the character of B, then character of B_t is $\psi_t(u) = \psi(tut^{-1})$. We say that the representations B and B_t or their characters ψ and ψ_t are conjugate with respect to G. Also

$$\begin{aligned}\langle \psi_t, \psi_t \rangle_H &= \frac{1}{h} \sum_{u \in H} \psi_t(u) \psi_t(u^{-1}) \\ &= \frac{1}{h} \sum_{u \in H} \psi(tut^{-1}) \psi(tu^{-1}t^{-1}) \\ &= \frac{1}{h} \sum_{z \in H} \psi(z) \psi(z^{-1}),\end{aligned}$$

where $z = tut^{-1}$ runs over H when u does. Therefore $\langle \psi_t, \psi_t \rangle_H = \langle \psi, \psi \rangle_H$. So if ψ is a simple character of H, then so is each ψ_t.

Theorem 45. *Let H be a normal subgroup of G and suppose that ξ is a simple character of H. Then ξ^G is a simple character of G if and only if $\xi_t \neq \xi$, whenever $t \notin H$.*

Proof. We know that if ξ is a simple character of H and if $t \in H$, then $\xi_t(u) = \xi(u)$ for $u \in H$ (since u and tut^{-1} are conjugate elements, $\xi_t(u) = \xi(tut^{-1}) = \xi(u)$). Let W be the set of those elements w of G for which

$$\xi(wuw^{-1}) = \xi(u) \tag{3.12}$$

Then W is a subgroup of G. Let $w_1, w_2 \in G$. Then $\xi(w_1 u w_1^{-1}) = \xi(u) = \xi(w_2 u w_2^{-1})$ for $u \in H$. Then

$$\begin{aligned}\xi(w_2^{-1} w_1 u (w_2^{-1} w_1)^{-1}) &= \xi(w_2^{-1}(w_1 u w_1^{-1}) w_2) \\ &= \xi(w_1 u w_1^{-1}) \\ &= \xi(u).\end{aligned}$$

Also, $H \leq W \leq G$. Suppose that $\xi_p = \xi_q$; that is,

$$\xi(pup^{-1}) = \xi(quq^{-1}). \quad (u \in H) \tag{3.13}$$

Replacing u by $q^{-1}uq$ (since $q^{-1}uq \in H$), (3.13) becomes $\xi(pq^{-1}uqp^{-1}) = \xi(u) \quad (u \in H)$
$\implies pq^{-1} \in W$; i.e. $Wp = Wq$. Thus the number of distinct conjugates of ξ is equal to the number of cosets of W in G; i.e. if

$$G = Wt_1 \cup Wt_2 \cup \cdots \cup Wt_r, \tag{3.14}$$

where $t_i = 1$ is a coset decomposition of G with respect to W, then $\xi_{t_1}(= \xi), \xi_{t_2}, \cdots, \xi_{t_r}$ is a complete set of distinct conjugates of ξ. Now consider the induced character ξ^G. $\xi^G(x) = \frac{1}{h}\sum_{y\in G}\xi(yxy^{-1})$. Using (3.14), $\xi^G(x) = \frac{1}{h}\sum_{w\in W}\sum_{j=1}^{r}\xi(wt_jxt_j^{-1}w^{-1})$. Restricting this equation to H,

$$\begin{aligned}\xi_H^G(u) &= \frac{1}{h}\sum_{w\in W}\sum_{j=1}^{r}\xi(wt_jut_j^{-1}w^{-1}) \\ &= \frac{1}{h}\sum_{w\in W}\sum_{j=1}^{r}\xi(t_jut_j^{-1}).\end{aligned}$$

Since the terms are independent of w,

$$\xi_H^G(u) = \frac{|w|}{h}\sum_{j=1}^{r}\xi_{t_j}(u) = [W:H]\sum_{j=1}^{r}\xi_{t_j}(u). \tag{3.15}$$

Since each $\xi_{t_j}(u)$ is a simple character, the above expression must be the unique Fourier analysis of ξ_H^G. By reciprocity law,

$$\langle \xi^G, \xi^G\rangle_G = \langle \xi_H^G, \xi\rangle_H \tag{3.16}$$

On taking the inner product of (3.15) with ξ, we get $\langle \xi_H^G, \xi \rangle_H = [W : H]$ (since simple characters are orthogonal and $\xi = \xi_{t_1}$); i.e. $\langle \xi^G, \xi^G \rangle_G = [W : H]$ from (3.16). Hence ξ^G is a simple character if and only if $W = H$; i.e. ξ^G is a simple character if and only if $\xi_t \neq \xi$ whenever $t \notin H$. □

Theorem 46. *(Clifford's Theorem)*
Let H be a normal subgroup of G, and let χ be a simple character of G. Then there exists a simple character ξ of H such that $\chi_H = e \sum_{j=1}^{r} \xi_{t_j}$ where e is a positive integer and the sum involves a complete set of conjugates of ξ.

Proof. Let χ_H be expressed as a linear combination of the simple characters of H. Then at least one non-zero term must occur. Choose the notation in such a way that the coefficient of ξ does not vanish; i.e. $\langle \xi, \chi_H \rangle_H = e > 0$. By reciprocity law, $\langle \xi^G, \chi \rangle_G = e$.
So the Fourier analysis of ξ^G is

$$\xi^G = e\chi + e'\chi' + e''\chi'' + \cdots, \tag{3.17}$$

where $\chi, \chi', \chi'', \ldots$ are simple characters of G and $e' \geq 0, e'' \geq 0, \ldots$. On restricting (3.17) to H,

$$\xi_H^G = e\chi + e'\chi'_H + e''\chi''_H + \cdots \tag{3.18}$$

Let η be a simple character of H that is not of the form ξ_{t_j}. Then $\langle \xi_H^G, \eta \rangle_H = 0$. (since $\xi_H^G(u) = \dfrac{|w|}{h} \sum_{j=1}^{r} \xi_{t_j}(u)$ is the Fourier analysis of ξ_H). Taking inner products with η on both sides of (3.18), $0 = e\langle \chi_H, \eta \rangle_H + e'\langle \chi'_H, \eta \rangle_H + \cdots$. Since

the inner product of any two characters is always a non-negative integer, each term must be zero and so $\langle \chi_H, \eta \rangle = 0$ (since $e \neq 0$).

Hence the Fourier analysis of χ_H does not involve η and hence is a linear combination of η and its conjugates say,

$$\chi_H = \sum_{j=1}^{r} b_j \xi_{t_j} \tag{3.19}$$

where $b_1 = \langle \chi_H, \xi_{t_1} \rangle_H = \langle \chi_H, \xi \rangle_H = e$. Now we show that all the coefficients in (3.19) equals b_i. Writing $t = t_j$,

$$\begin{aligned} \langle \xi_t, \chi_H \rangle_H &= \frac{1}{h} \sum_{u \in H} \xi_t(u) \overline{\chi}(u) \\ &= \frac{1}{h} \sum_{u \in H} \xi_t(tut^{-1}) \overline{\chi}(u). \end{aligned}$$

Replacing the summation variable by $v = tut^{-1}$,

$$\begin{aligned} \langle \xi_t, \chi_H \rangle_H &= \frac{1}{h} \sum_{v \in H} \xi(v) \overline{\chi}(t^{-1}vt) \\ &= \frac{1}{h} \sum_{v \in H} \xi(v) \overline{\chi}(v) \\ &= \langle \xi, \chi_H \rangle_H = e. \end{aligned}$$

Hence (3.19) become $\chi_H = e \sum_{j=1}^{r} \xi_{t_j}$. □

Chapter 4

Permutation Groups

4.1 Transitive group

Let G be a permutation group of degree n, acting on the symbols $1, 2, \ldots, n$. Thus a typical element of G is a permutation $x \to \alpha x$ $(\alpha = 1, 2, \ldots, n)$.

Definition 47. *The group G is said to be* ***transitive*** *on $1, 2, \ldots, n$ if, for every pair (α, β), where $1 \leq \alpha, \beta \leq n$, there exists at least one permutation $x_{\alpha\beta}$ in G with the property that $\alpha x_{\alpha\beta} = \beta$.*
α and β need not be distinct and $x_{\alpha\beta}$ is not required to be unique.

Criterion. The group G is transitive if and only if for each α $(1 \leq \alpha \leq n)$, there exists at least one permutation p_α in G with the property that $1p_\alpha = \alpha$.

Proof. If G is transitive, for every pair (α, β), $1 \leq \alpha, \beta \leq n$, there exists a permutation $x_{\alpha\beta}$ in G such that $\alpha x_{\alpha\beta} = \beta$. Taking $\alpha = 1, \beta = \alpha$, we get $1x_\alpha = \alpha$ and renaming $x_\alpha = p_\alpha$, the criterion is true.
Conversely, if p_α satisfies $1p_\alpha = \alpha$, then $\alpha(p_\alpha^{-1}p_\beta) = \beta$. So taking $x_{\alpha\beta} = p_\alpha^{-1}p_\beta$, the group is transitive. □

Example 48. *Consider the following subgroups of* S_4 *:*
$A : 1,\ (12)(34),\ (13)(24),\ (14)(23),$
$B : 1,\ (12), (34), (12)(34).$

The criterion tells us that A is transitive. But B is clearly not transitive, as it contains no permutation which maps 1 into 3. But we have $A \cong B$ as abstract groups.

Definition 49. *Those permutations of* G *which leave the symbol* 1 *fixed form a subgroup of* G*, which is called the* ***stabiliser*** *of* 1 *(in* G*).*

Similarly we can define the stabiliser of α, $\alpha \in G$.

Proposition 50. *Let* G *be a transitive group on* $1, 2, \ldots, n$ *and suppose that* $1p_\alpha = \alpha$ $(\alpha = 1, 2, \ldots, n)$*. If* H *is the stabiliser of* 1*, then the stabiliser of* α *is the group* $p_\alpha^{-1}Hp_\alpha$*, which is isomorphic to* H*.*

Proof. Let H and H_α be the stabilisers of the symbols 1 and α respectively. First, let u be an arbitrary element of H, that is $1u = 1$.
$\implies p_\alpha^{-1}up_\alpha = 1 \implies \alpha(p_\alpha^{-1}up_\alpha) = \alpha \implies p_\alpha^{-1}up_\alpha \in H_\alpha$

$\implies p_\alpha^{-1} H p_\alpha \subseteq H_\alpha$, the stabiliser of α.
Conversely, suppose that $v \in H_\alpha$; i.e. $\alpha v = \alpha$. Since $1p_\alpha = \alpha$, $\alpha v = \alpha \implies 1p_\alpha v = 1p_\alpha \implies 1(p_\alpha v p_\alpha^{-1}) = 1$.
Hence $p_\alpha v p_\alpha^{-1} \in H$, the stabiliser of 1.
$\implies p_\alpha H_\alpha p_\alpha^{-1} \subseteq H \implies H_\alpha \subseteq p_\alpha^{-1} H p_\alpha$.
This proves that $H_\alpha = p_\alpha^{-1} H p_\alpha$, which is clearly isomorphic to H. □

So we assume that G is transitive and that the permutations p_α $(\alpha = 1, 2, \ldots, n)$ have been chosen.
Then the permutation x transforms 1 into α if and only if x lies in the coset Hp_α, where H is the stabiliser of 1.

Proof. Let $1x = \alpha$. We know that $1p_\alpha = \alpha$. Thus $x = p_\alpha$ which implies $x \in Hp_\alpha$.
Conversely, suppose that $x \in Hp_\alpha$; i.e. $x = up_\alpha$ for some $u \in H$.
Hence $1x = 1(up_\alpha) = (1u)p_\alpha = 1p_\alpha = \alpha$. □

As α varies from 1 to n we obtain all the permutations of G. Thus we have the coset decomposition $G = Hp_1 \cup Hp_2 \cup \cdots \cup Hp_n$. It follows that $[G : H] = n$; i.e. the degree of a transitive group divides the order of the group.

Proposition 51. *Let G be a transitive permutation group of order g. If $\nu(x)$ is the natural character of G, then $\Sigma_{x \in G} \nu(x) = g$.*

Proof. Every permutation group has a natural representation in terms of permutation matrices. Denoting this rep-

resentation by $N(x)$ and its trace by $\nu(x)$, we have $\nu(x) =$ number of symbols fixed by x. We denote the stabiliser of 1 by H. Let $\xi^{(1)}$ be the trivial character of H. We assert that ${\xi^{(1)}}^G(x) = \Sigma_{\alpha=1}^{n} \xi^{(1)}(p_\alpha x p_\alpha^{-1}) = \nu(x)$.

For each term in the sum is either zero or unity, and it is equal to unity if and only if $p_\alpha x p_\alpha^{-1} \in H$, that is $x \in p_\alpha^{-1} H p_\alpha = H_\alpha$, which means that x leaves α fixed. Thus the sum counts the number of symbols fixed by x $(\nu(x))$, which proves our assertion.

If $\chi^{(1)}$ is the trivial character of G, then its restriction to H is the trivial character of H, that is $\chi_H^{(1)} = \xi^{(1)}$, which gives $\langle \chi_H^{(1)}, \xi^{(1)} \rangle_H = 1$.

On applying the Reciprocity Theorem to the last equation we get

$\langle \chi^{(1)}, {\xi^{(1)}}^G \rangle_G = 1.$

i.e. $\langle \chi^{(1)}, \nu \rangle_G = 1.$

i.e. $\dfrac{1}{g} \Sigma_{x \in G} \chi^{(1)}(x^{-1}) \nu(x) = 1.$

i.e. $\dfrac{1}{g} \Sigma_{x \in G} \nu(x) = 1$, which proves the proposition. □

Definition 52. *A permutation group G, acting on the symbols $1, 2, \ldots, n$ is said to be **doubly transitive** if corresponding to any two pairs of symbols (α, β) and (γ, δ), where $\alpha \neq \beta$ and $\gamma \neq \delta$, there exists at least one permutation x in G satisfying $\alpha x = \gamma$, $\beta x = \delta$.*

Theorem 53. *If G is a doubly transitive group with natural character $\nu(x)$, then the function $\nu(x) - 1$ is a simple character of G.*

Proof. Suppose that G is doubly transitive and let H be the stabiliser of 1 in G. A typical permutation of H has the form $u = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix}$. We associate with u the permutation $u' = \begin{pmatrix} 2 & 3 & \cdots & n \\ \alpha_2 & \alpha_3 & \cdots & \alpha_n \end{pmatrix}$.

As u ranges over H, u' ranges over a group H' acting on the $n-1$ elements $2, 3, \ldots, n$. Thus the correspondence $u \leftrightarrow u'$ establishes an isomorphism between H and H'.

We claim that H' is transitive. For if β and γ are any symbols, other than 1, we know that there exists a permutation v in G with the properties that $1v = 1$ and $\beta v = \gamma$ (taking the pairs $(1, \beta)$ and $(1, \gamma)$ in the definition). $1v = 1$ implies v belongs to H and, using the correspondence between u and u', we get $\beta v' = \gamma$. Thus H' is transitive.

Evidently, $\nu(u) = 1 + \nu'(u')$, where ν' is the natural character of H'. Summing over u, or equivalently over u', $\Sigma_{u \in H} \nu(u) = h + \Sigma_{u' \in H'} \nu'(u')$ where $h = |H| = |H'|$.

Since H' is transitive, by the above proposition, $\Sigma_{u' \in H'} \nu'(u') = h$. Thus $\Sigma_{u \in H} \nu(u) = 2h$.

Hence $\langle \nu_H, \xi^{(1)} \rangle_H = \frac{1}{h} \Sigma_{u \in H} \nu(u) \xi^{(1)}(u^{-1}) = \frac{1}{h} \Sigma_{u \in H} \nu(u) = 2$.

Using the reciprocity theorem, $\langle \nu, {\xi^{(1)}}^G \rangle_G = 2$.

We have proved that ${\xi^{(1)}}^G = \nu$. Thus $\langle \nu, \nu \rangle_G = 2$. Since G is transitive, $\Sigma_{x \in G} \nu(x) = g$. So in the Fourier analysis of ν,

$$e_1 = \langle \nu, \chi^{(1)} \rangle = \frac{1}{g} \Sigma_{x \in G} \nu(x) \chi^{(1)}(x^{-1}) = \frac{1}{g} \Sigma_{x \in G} \nu(x) = 1.$$

Hence the Fourier analysis of ν is given by $\nu = \chi^{(1)} +$

$e_2\chi^{(2)} + \cdots$. Taking inner product with ν on both sides, $\langle \nu, \nu \rangle = \langle \nu, \chi^{(1)} \rangle + e_2^2 + e_3^2 + \cdots$; that is, $2 = 1 + e_2^2 + e_3^2 + \cdots$. This means that precisely one of the coefficients $e_2, e_3, \ldots$ is equal to unity, all the others being zero. Hence $\nu - \chi^{(1)}$ is a simple character. □

Example 54. *Construct the character table of the symmetric group S_4.*

The conjugacy classes of S_4 are represented by $1, (12), (123), (12)(34), (1234)$, comprising $1, 6, 8, 3, 6$ elements respectively. Hence S_4 possesses five irreducible representations. We denote the trivial character by $\chi^{(1)}$. The alternating character is $\chi^{(2)} : 1, -1, 1, 1, -1$, because the elements (12) and (1234) are odd while 1, (123) and $(12)(34)$ are even.

S_4 has the normal subgroup V : $1, (12)(34), (13)(24), (14)(23)$. With a convenient choice of coset representatives we may assume that $S_4/V : V, V(12), V(13), V(23), V(123), V(132)$.

Claim: S_4/V is non-abelian.

$V(12)V(13) = V(12)(13) = V(132)$ and $V(13)V(12) = V(13)(12) = V(123)$.

Also, $|S_4/V| = 6$. Since S_3 is the smallest non-abelian group of order 6, $S_4/V \cong S_3$. Hence any one of the characters of S_3 may be lifted so as to yield a character of S_4 by lifting process. The only character that gives a useful result is the character of degree two in the Table 2.3,

which is $\chi^{(3)}$. *In* S_4/V, $C_1 : V, C_2 : V(12), C_3 : V(123)$. *So* $\chi_0^{(3)}(V) = 2, \chi_0^{(3)}(V(12)) = 0, \chi_0^{(3)}(V(123)) = -1$. *Lifting these characters to* S_4, $\chi^{(3)}(1) = \chi^{(3)}((12)(34)) = 2$, $\chi^{(3)}(123) = -1, \chi^{(3)}(12) = 0$. $(1234) = (12)(34)(13) \in V(13) = V(12)$. *So* $\chi^{(3)}(1234) = \chi_0^{(3)}(V(12)) = 0$.

Thus $\chi^{(3)} : 2, 0, -1, 2, 0$.

Since S_4 *is the group of all permutations on* 4 *symbols,* S_4 *is doubly transitive. So if* $\nu(x)$ *is the natural character of* S_4, *then* $\nu(x)-1$ *is also a simple character of* S_4. *Let* $\chi^{(4)} = \nu - 1$. $\nu(1) = 4, \nu(12) = 2, \nu(123) = 1, \nu((12)(34)) = 0 = \nu(1234)$ *(since* $\nu(x)$ *is the number of elements of* S_4 *fixed by* x*).*

Thus $\chi^{(4)} : 3, 1, 0, -1, -1$.

Finally, we can see that $\chi^{(5)} = \chi^{(2)}\chi^{(4)}$ *is another simple character. Since* $\chi^{(5)}(x) = \chi^{(2)}(x)\chi^{(4)}(x)$,

$\chi^{(5)} : 3, -1, 0, -1, 1$, *so that*

$$\begin{aligned}\langle \chi^{(5)}, \chi^{(5)} \rangle &= \frac{1}{24}\Sigma_{x \in S_4}\chi^{(5)}(x)\chi^{(5)}(x^{-1}) \\ &= \frac{1}{24}\Sigma_{\alpha=1}^{5}\Sigma_{x \in C_\alpha}[\chi^{(5)}(x)]^2 \\ &= \frac{1}{24}\Sigma_{\alpha=1}^{5} h_\alpha [\chi^{(5)}(x)]^2 \\ &= \frac{1}{24}[1 \times 3^2 + 6 \times (-1)^2 + 8 \times 0^2 + 3 \times (-1)^2 + 6 \times 1^2] \\ &= 1.\end{aligned}$$

Thus the character table of S_4 *is*

S_4 :

C_α	1	(12)	(123)	(12)(34)	(1234)
h_α	1	6	8	3	6
$\chi^{(1)}$	1	1	1	1	1
$\chi^{(2)}$	1	−1	1	1	−1
$\chi^{(3)}$	2	0	−1	2	0
$\chi^{(4)}$	3	1	0	−1	−1
$\chi^{(5)}$	3	−1	0	−1	1

Table 4.1:

4.2 The Symmetric Group

Each permutation of S_n can be resolved into a product of disjoint cycles in a unique manner except for the order of the cycle factors. A cycle involving a single symbol indicates that this symbol remains fixed. Two elements of S_n are conjugate if and only if they have the same cycle pattern. For example, in S_6, the permutations $x = (146)(35)(2)$ and $y = (243)(16)(5)$ have the same cycle pattern and are therefore conjugate; we can see that $t^{-1}xt = y$, where $t = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 1 & 4 & 6 & 3 \end{pmatrix}$.
Thus each conjugacy class C_α of S_n is determined by its cycle pattern comprising, say, α_1 cycles of degree 1, α_2 cycles of degree 2, and so on. Accordingly, the specification of C_α will be described by the formula

$$\alpha : \alpha_1 + 2\alpha_2 + 3\alpha_3 + ... + n\alpha_n = n, \tag{4.1}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are non-negative integers. In the above example, for the conjugacy class containing x and y, $\alpha_1 = \alpha_2 = \alpha_3 = 1$.

Alternatively, each permutation of C_α is the product of, say, u cycles whose degrees, in some order, are $p_1, p_2, \ldots, p_n$, respectively. Hence C_α is determined by the partition

$$p : p_1 + p_2 + \cdots + p_u = n, \tag{4.2}$$

where the parts are positive integers. (In the above example, $u = 3, p_1 = 3, p_2 = 2, p_3 = 1$ and $p_1 + p_2 + p_3 = 6$.)

We can add $n - u$ zero terms and to define a partition of n as a sum of n non-negative integers, thus $p : p_1 + p_2 + \cdots + p_n = n$, which is abbreviated to $|p| = n$. In order to achieve uniqueness we may impose the conditions

$$p_1 \geq p_2 \geq \cdots \geq p_n \geq 0. \tag{4.3}$$

Let k be the number of conjugacy classes of S_n. Then k is the number of solutions of (4.1) or the number of solutions of (4.2) and (4.3).

When C_α is specified by (4.1),

$$|C_\alpha| = h_\alpha = \frac{n!}{1^{\alpha_1}\alpha_1!2^{\alpha_2}\alpha_2! \ldots n^{\alpha_n}\alpha_n!}. \tag{4.4}$$

So in the above example, $h_\alpha = \dfrac{6!}{1^1 1! 2^1 1! 3^1 3!}$.

Note. *In S_n each element is conjugate to its inverse, since*

x and x^{-1} possess the same cycle pattern. Hence if ϕ is a character of S_n we have that $\phi(x) = \phi(x^{-1}) = \overline{\phi}(x)$. It follows that all characters of S_n have real values.

4.3 Induced Characters of S_n

We aim to compute the simple characters $\chi^{(1)}, \chi^{(2)}, \ldots, \chi^{(k)}$ of S_n. Corresponding to an arbitrary partition $p : p_1 + p_2 + \cdots + p_n = n$, we form the subgroup $H_p = S_{p_1} \times S_{p_2} \times \cdots \times S_{p_n}$. Here S_{p_1} acts on the symbols $1, 2, \ldots, p_1$; S_{p_2} acts on the symbols $p_1 + 1, \ldots, p_1 + p_2$; S_{p_3} acts on the symbols $p_1 + p_2 + 1, \ldots, p_1 + p_2 + p_3$, and so on, with the understanding that S_{p_j}, is the unit group when $p_j = 0$. Then $|H_p| = p_1!p_2! \cdots + p_n!$. Let ϵ be the trivial character of H_p, and let ϕ_p be the character of S_n induced by ϵ. Then its value on C_α is given by

$$\phi_\alpha^{(p)} = \frac{1}{h_\alpha}[S_n : H_p]\Sigma_{z \in C_\alpha \cap H_p}\epsilon(z). \tag{4.5}$$

Hence $\phi_\alpha^{(p)} = \frac{1}{h_\alpha}[S_n : H_p]|C_\alpha \cap H_p|$.

Let $u = u_1 u_2 \ldots u_n$ be a typical element of H_p where $u_j \in S_{p_j}$ $(j = 1, 2, \ldots, n)$, and suppose that u_j is the product of α_{j1} 1-cycles, α_{j2} 2-cycles, α_{j3} 3-cycles, ... We determine the conditions on the integers α_{ij} which ensure that u belongs to C_α.

First, the total number of i-cycles must add up to α_i. Thus $A : \alpha_i = \alpha_{1i} + \alpha_{2i} + \cdots + \alpha_{ni}$, $(i = 1, 2, \ldots, n)$.

Secondly, u_i involves p_j symbols, that is $B : p_j = \alpha_{j1} + 2\alpha_{j2} + \cdots + n\alpha_{jn}$ $(j = 1, 2, \ldots, n)$. In the equations A and B, the

numbers α_i and p_j are prescribed, and the matrix (α_{ji}) is unknown.

Next, we ask how many elements of $C_\alpha \cap H_p$ correspond to a particular solution (α_{ji}). For a fixed value of j, the equation B specifies a conjugacy class of S_{p_j}. By (4.4), this class contains $h^{(i)}_{\alpha_{ji}} = \dfrac{p_j!}{1^{\alpha_{j1}}\alpha_{j1}!2^{\alpha_{j2}}\alpha_{j2}!\ldots n^{\alpha_{jn}}\alpha_{jn}!}$ elements.

Letting j range from 1 to n we obtain $\Pi_{j=1}^n h^{(i)}_{\alpha_{ji}}$ elements of $C_\alpha \cap H_p$, all corresponding to the same matrix (α_{ji}). Hence $|C_\alpha \cap H_p| = \Sigma_{A,B}\Pi_{j=1}^n h^{(i)}_{\alpha_{ji}}$, where the summation is extended over all solutions of A and B. Substituting this expression in (4.5) we get $\phi^{(p)}_\alpha = \Sigma_{A,B}\Pi_{j=1}^n \dfrac{\alpha_j!}{\alpha_{j1}!\alpha_{j2}!\ldots\alpha_{jn}!}$. Thus for each partition we have constructed a compound character of S_n. The number of these compound characters is equal to the number of simple characters.

4.4 Exercise

1. After a jolly party each reveller picks up at random one of the umbrellas which had previously been deposited in the cloakroom. Prove that only one gentleman is expected to return with his own umbrella.

Solution

1. Suppose there are n members in the party. Let $\nu(x)$ be the number of fixed objects under the permutation x, that is the number of gentlemen who pick

up their own umbrellas. Then the average value of $\nu(x)$ is $\frac{1}{n!}\Sigma_{x\in S_n}\nu(x) = \frac{1}{n!}n! = 1$ (as $\Sigma_{x\in G}\nu(x) = g, \Sigma_{x\in S_n}\nu(x) = n!$).

Syllabus

SEMESTER 4(Elective)

MTH4E12: REPRESENTATION THEORY

No. of Credits: 3

No. of hours of Lectures/week : 5

TEXT : Walter Ledermann, Introduction to Group Characters(Second Edition).

Module 1

Introduction, G- modules, Characters, Reducibility, Permutation Representations, Complete reducibility, Schurs lemma, The commutant(endomorphism) algebra. (Sections: 1.1 to 1.8)

Module 2

Orthogonality relations, the group algebra, the character table, finite abelian groups, the lifting process, linear characters. (section: 2.1 to 2.6)

Module 3

Induced representations, reciprocity law, the alternating group A_5, Normal sub- groups, Transitive groups, the symmetric group, induced characters of S_n. (Sections: 3.1 to 3.4, 4.1 to 4.3)

References:

[1] C. W. Kurtis and I. Reiner: Representation Theory of

Finite Groups and Associative Algebras, John Wiley and Sons, New York(1962)

[2] Faulton: The Representation Theory of Finite Groups, Lecture Notes in Mathematics, No. 682, Springer 1978.

[3] C. Musli: Reprsentations of Finite Groups, Hindustan Book Agency, New Delhi (1993).

[4] I. Schur: Theory of Group Characters, Academic Press, London (1977).

[5] J.P. Serre: Linear Representation of Finite Groups, Graduate Text in Mathematics, Vol 42, Springer (1977).